MATH HL - OPTION CALCULUS
REVISION
(one example for each case)
by Christos Nikolaidis

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<td>1. L’ Hôpital</td>
<td>It works only for fractions $\frac{0}{0}$ or $\frac{\infty}{\infty}$</td>
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**Solution**

$$\lim_{x \to 0} \frac{e^x - 1 - x - 0.5x^2}{x^3}$$

$$\lim_{x \to 0} \frac{3e^x - 1 - x - 0.5x^2}{3x^2} \cdot \frac{\lim_{x \to 0} \frac{3e^x - 1 - x - 0.5x^2}{3x^2}}{x} = \frac{3e^0 - 1 - 0 - 0.5(0)^2}{3(0)^2}$$

$$= \frac{3 - 1}{0} = \frac{2}{0}$$

2. Continuity - Differentiability

**Let**

$$f(x) = \begin{cases} 
  x^2 + 4, & x \leq 2 \\
  ax, & 2 < x \leq 3 \\
  b - cx^2, & x > 3 
\end{cases}$$

Find the values of $a$, $b$, $c$ given that the function is continuous and differentiable.

**Solution**

- $f$ is continuous at $x=2$, so $\lim_{x \to 2}(x^2+4) = \lim_{x \to 2}ax = f(2)$
  Thus, $8=2a=8 \Rightarrow a=4$

- $f$ is continuous at $x=3$, so $\lim_{x \to 3}ax = \lim_{x \to 3}(b-cx^2) = f(3)$
  Thus $12=b-9c=12 \Rightarrow b-9c=12 (*)$

- $f$ must be differentiable at $x=2$. Indeed, for $x=2$
  $f'(2) = (x^2+4)' = 2x = 4$, $f'(2) = (ax)' = a = 4$

- $f$ must be differentiable at $x=3$. That is, for $x=3$
  $f'(3) = (ax)' = a = 4$, $f'(3) = (b-cx^2)' = -2cx = -6c$
  so that $4 = -6c$ (**)

$\Rightarrow (*)$ and $(**)$ give $c=-2/3$ and $b=6$

Therefore $a=4$, $b=6$, $c=-2/3$
3. Rolle Theorem – Mean Value Theorem (MVT)

Let \( f(x) = \sin x \)

(a) Explain why Rolle’s Thm holds in the interval \([0, \pi]\) and find the corresponding value of \( c \)
(b) Does it hold in the interval \([0.1, \pi - 0.1]\)?
(c) Apply the Mean Value Theorem in \([0, \pi/2]\) and find the corresponding value of \( c \)

**Solution**

(a) \( f(x) = \sin x \) is continuous and differentiable in \([0, \pi]\) and \( f(0) = f(\pi) = 0 \)

Thus there is some value \( c \in ]0, \pi[ \) s.t. \( f'(c) = 0 \)

Indeed, \( f(x) = \cos x \) and \( \cos c = 0 \) for \( c = \pi/2 \)

(b) \( f(x) \) is still continuous/differentiable in \([0.1, \pi - 0.1]\)

\( f(0.1) = f(\pi - 0.1) \)

Again \( c = \pi/2 \)

(c) \( f(x) \) is continuous and differentiable in \([0, \pi/2]\)

Thus there is some value \( c \in ]0, \pi/2[ \) such that

\[
 f'(c) = \frac{f(\pi/2) - f(0)}{\pi/2 - 0}, \text{ i.e. } \cos c = \frac{2}{\pi}
\]

Indeed the GDC gives \( c = 0.881 \)

Let \( f(x) = x^2 + 5x - 7 \)

Use Rolle’s Thm to show that \( f(x) \) has only one root

**Solution**

\( f(1) = -1 < 0 \) and \( f(2) = 11 > 0 \)

Thus there is at least one root between 1 and 2

Suppose that there are two roots \( a \) and \( b \)

Then
- \( f(x) \) is continuous and differentiable in \([a, b]\)
- \( f(a) = f(b) \)

Thus, by Rolle,

there is a value \( c \in [a, b] \) such that \( f'(c) = 0 \)

But, \( f'(x) = 3x^2 + 5 \) which has no roots which contradicts the existence of \( c \)

Therefore there is only one root.

The minimum requirement is:
- Continuous in \([a, b]\)
- Differentiable in \([a, b]\)
- \( f(a) = f(b) \)

Then \( f'(c) = 0 \) for some \( c \in [a, b] \)

The minimum requirement is:
- Continuous in \([a, b]\)
- Differentiable in \([a, b]\)

Then \( f'(c) = \frac{f(b) - f(a)}{b - a} \)

for some \( c \in [a, b] \)

A polynomial is continuous and differentiable everywhere

In general, if \( f(x) \) has \( n \) distinct roots, then \( f(x) \) must have at least \( n - 1 \) roots (apply Rolle between any two consecutive roots)
Use the MVT for \( f(x) = e^x \) to show that \( e^x \geq x + 1 \)

**Solution**

Consider \( f(x) = e^x \)

It is continuous and differentiable everywhere.

We apply the theorem in the interval \([0, x]\) where \( x > 0 \)

\[
f'(c) = \frac{e^x - e^0}{x - 0} \quad \text{for some } c \in [0, x]
\]

That is \( \frac{e^x - 1}{x} > 1 \) since \( x \) is positive

We also apply in the interval \([x, 0]\) where \( x < 0 \)

\[
f'(c) = \frac{e^x - e^0}{x - 0} \quad \text{for some } c \in [x, 0]
\]

That is \( \frac{e^x - 1}{x} < 1 \) since \( x \) is negative

Finally, for \( x = 0 \), \( e^x = x + 1 \)

Therefore, in any case \( e^x \geq x + 1 \)

---

4. **Fundamental Theorem of Calculus**

Find

(a) \( \int_{0}^{2} \left( \frac{3x^2 + x - 2}{x^2 + x + 1} \right) \, dx \)

(b) \( \frac{d}{dx} \left[ \int_{0}^{x} (\ln t)^3 \, dt \right] \)

(c) \( \frac{d}{dx} \left[ \int_{x}^{2} (\ln t)^3 \, dt \right] \)

(d) \( \frac{d}{dx} \left[ \int_{x}^{2} (\ln t)^3 \, dt \right] \)

**Solution**

(a) \( \int_{0}^{2} \left( \frac{3x^2 + x - 2}{x^2 + x + 1} \right) \, dx = \left[ \frac{3x^2 + x - 2}{x^2 + x + 1} \right]_{0}^{1} = \frac{2}{3} - \left[ -\frac{8}{3} \right] = \frac{3}{2} \)

(b) \( \frac{d}{dx} \left[ \int_{0}^{x} (\ln t)^3 \, dt \right] = (\ln x)^3 \)

(c) \( \frac{d}{dx} \left[ \int_{x}^{2} (\ln t)^3 \, dt \right] = 2x(\ln x)^3 \)

(d) \( \frac{d}{dx} \left[ \int_{x}^{2} (\ln t)^3 \, dt \right] = \frac{d}{dx} \left[ \int_{0}^{x} (\ln t)^3 \, dt - \int_{0}^{x} (\ln t)^3 \, dt \right] = 2x(\ln x)^3 - (\ln x)^3 \)

- \( \int_{a}^{b} f(x) \, dx = [f(x)]_{a}^{b} = f(b) - f(a) \)
- \( \frac{d}{dx} \left[ \int_{a}^{x} f(t) \, dt \right] = f(x) \)
- \( \frac{d}{dx} \left[ \int_{a}^{b} f(t) \, dt \right] = f(g(x)) \cdot g'(x) \)

use \( \frac{d}{dx} \left[ \int_{a}^{b} f(t) \, dt \right] = \frac{d}{dx} \left[ \int_{a}^{b} f(t) \, dt \right] \)
5. Riemann Sum

Consider \( \int_0^\pi \sin x \, dx \). Find the Riemann sum

(a) if you consider 4 subintervals of equal length and as representative values \( x_i \)
(\( i \) the minima (ii) the maxima

(b) if you consider 2 subintervals of equal length and the midpoints as representative values

(c) Compare with the actual value of the integral

Solution

(a) Subintervals \([0, \pi/4] \quad [\pi/4, \pi/2] \quad [\pi/2, 3\pi/4] \quad [3\pi/4, \pi]\]

(i) R.S. = \( f(0) \frac{\pi}{4} + f(\pi/4) \frac{\pi}{4} + f(\pi/2) \frac{\pi}{4} + f(3\pi/4) \frac{\pi}{4} = 1.11 \)

(ii) R.S. = \( f(\pi/4) \frac{\pi}{4} + f(\pi/2) \frac{\pi}{4} + f(2\pi/4) \frac{\pi}{4} + f(3\pi/4) \frac{\pi}{4} = 2.68 \)

(b) Subintervals \([0, \pi/2] \quad [\pi/2, \pi]\]

R.S. = \( f(\pi/2) \frac{\pi}{2} + f(3\pi/4) \frac{\pi}{2} = 2.22 \)

(c) \( \int_0^\pi \sin x \, dx = 2 \)

So the approximation in (b) is much better despite the larger subintervals

6. Improper Integrals

Find \( (a) \int_1^b \frac{1}{x} \, dx \), \( (b) \int_{-\infty}^{1/b} \frac{1}{x^2} \, dx \), \( (c) \int_{-\infty}^{\infty} e^{-x} \, dx \)

Solution

(a) \( \lim_{b \to \infty} \int_1^b \frac{1}{x} \, dx = \lim_{b \to \infty} \ln |x| \bigg|_1^b = \lim_{b \to \infty} |\ln b - 0| = +\infty \)

(b) \( \lim_{b \to \infty} \int_1^b \frac{1}{x^2} \, dx = \lim_{b \to \infty} \frac{-1}{x} \bigg|_1^b = \lim_{b \to \infty} \left( \frac{-1}{b} + 1 \right) = 1 \)

(c) Find the indefinite integral first

\[ \int e^{-x} \, dx = -e^{-x} + \int e^{-x} \, dx = -e^{-x} + C \]

Thus,

\[ \int_0^b e^{-x} \, dx = \lim_{b \to \infty} \left[ -e^{-x} \right]_0^b = \lim_{b \to \infty} \left( -\frac{b+1}{e^b} + 1 \right) = 1 \]

It diverges

It converges to 1

It converges to 1
# 7. Differential Equations (D.E.)

## Solve \( \frac{dy}{dx} = x(y^2 + 1) \), if \( y=1 \) when \( x=0 \).

### Solution

\[
\frac{dy}{y^2 + 1} = xdx \implies \int \frac{dy}{y^2 + 1} = \int xdx
\]

\[
\arctan y = \frac{x^2}{2} + c
\]

For \( x=0, y=1 \) \( \implies \frac{\pi}{4} = c \)

Hence,

\[
\arctan y = \frac{x^2}{2} + \frac{\pi}{4}
\]

**Notice:** If they ask to express \( y \) in terms of \( x \):

**General solution:** \( y = \tan\left(\frac{x^2}{2} + c\right) \),

**Particular solution:** \( y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right) \)

## Solve \( \frac{dy}{dx} = \frac{36x^2 + 13xy + y^2}{x^2} \), if \( y=1 \) when \( x=1 \).

### Solution

\[
\frac{dy}{dx} = \frac{36x^2 + 13xy + y^2}{x^2} = \frac{dy}{dx} = \frac{36 + 13\frac{y}{x} + (\frac{y}{x})^2}{x^2}
\]

Let \( u = \frac{y}{x} \) \( \implies y = ux \) \( \implies \frac{dy}{dx} = u + x \frac{du}{dx} 

\[
\text{Hence,} \quad u + x \frac{du}{dx} = \frac{36 + 13u + u^2}{x} = \frac{36 + 13u + u^2}{x} = \frac{dy}{dx} = \frac{36 + 13u + u^2}{x}
\]

\[
\Rightarrow x \frac{du}{dx} = (u+6) \frac{dx}{x} \Rightarrow \frac{du}{u+6} = \frac{dx}{x} \Rightarrow \frac{u}{u+6} x = \ln |u+6| \Rightarrow u+6 = -\frac{\ln |x| - C}{\ln |u+6|}
\]

\[
\frac{u}{x} = -\frac{\ln |x| - C}{\ln |u+6|} - 6 \Rightarrow y = -\frac{x}{\ln |u+6|} - 6x
\]

\[
y(1) = 1 \Rightarrow -\frac{1}{\ln 7} - 6 = 1 \Rightarrow -\frac{1}{\ln 7} - 6 = 1 \Rightarrow \frac{1}{\ln 7} = C = -\frac{1}{\ln 7}
\]

Therefore,

\[
y = \frac{x}{\frac{1}{\ln 7} - 6x}
\]
Solution

\[ \frac{dy}{dx} + y \tan x = e^{3x} \cos x \]

**Integrating factor:**

\[ I = e^{ \int P(x) \, dx} = e^{ \int \tan x \, dx} = e^{\ln \cos x} = \frac{1}{\cos x} \]

Thus,

\[ Iy = \int IQ \, dx = y\left(\frac{1}{\cos x}\right) e^{3x} \]

\[ y = \left(\frac{e^{3x}}{3} + \frac{2}{3}\right) \cos x \]

The particular solution is

\[ y = \left(\frac{e^{3x}}{3} + \frac{2}{3}\right) \cos x \]

With Integrating Factor

as it can take the form

\[ \frac{dy}{dx} + P(x)y = Q(x) \]

[see booklet]

**Spot** \( P(x), Q(x) \)

[see booklet for \( I = e^{\int P(x) \, dx} \)]

Don’t put (+c) here

You have to remember that

\[ Iy = \int IQ \, dx \]

Euler

as it says so! 😊

Always in the form \( \frac{dy}{dx} = F(x, y) \)

[see booklet for formulas]

The GDC does it all!

Recursion

\[ a_{n+1} = a_n + 0.2 \]
\[ b_{n+1} = b_n + 0.2 (e^{3x} \cos x_n - b_n \tan x_n) \]

Set \( a_0 = 0 \), \( b_0 = 1 \)

**Notice:** the smaller step \( h \), the better approximation!
### 8. Isoclines – Slope Fields

Consider \( \frac{dy}{dx} = x + y \). In the following grid

(a) Sketch the isoclines
(b) Draw the Slope Field
(c) Sketch the particular solution passing through \((0,0)\)

**Solution**

(a) The isoclines are the curves \( x + y = c \).

That is \( y = -x + c \)

![Graph showing isoclines](image)

(b) We sketch the slope on each point

![Graph showing slope field](image)

(c) We draw a curve passing through \((0,0)\)

![Graph showing particular solution](image)

The grid is usually given!

If \( \frac{dy}{dx} = f(x,y) \)

the isoclines are \( f(x,y) = c \)

← that is all lines of slope -1

Slope 0 means
Slope 1 means
Slope -1 means

For example, at \((0,1)\) the slope is \( \frac{dy}{dx} = x + y = 1 \), that is

Notice that on each isocline the slopes are equal. E.g. on \( y = -x + 1 \) all slopes are 1.

Just look at the point \((0,0)\) and “follow the stream”.

Indeed, if you solve the d.e. (using integrating factor) you will find the particular solution \( y = e^x - x - 1 \) and the graph looks like that!
9. Series

\[ \sum_{n=1}^{\infty} \frac{2n+1}{3n^4+4} \]. Determine if it converges or diverges

Solution

It diverges since \( \lim_{n \to \infty} \frac{2n+1}{3n^4+4} = \frac{2}{3} \neq 0 \)

Divergence Test

[rarely used]
if the limit is not 0, we stop there! The series diverges!
If it is 0, we use another test.

\[ \sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)^2} \]. Determine if it converges or diverges

Solution

We check the corresponding integral \( I = \int_{2}^{\infty} \frac{1}{x \ln(x)^2} \, dx \)

\[ \int \frac{1}{x \ln(x)^2} \, dx = \int \frac{\ln(x)}{x^2} \, dx = -\frac{1}{\ln x} + c \]

Thus \( I = \lim_{b \to \infty} \left[ -\frac{1}{\ln x} \right]_a^b = \lim_{b \to \infty} \left[ -\frac{1}{\ln b} + \frac{1}{\ln a} \right] = \frac{1}{\ln a} \)

The integral converges so the series converges.

\[ \sum_{n=1}^{\infty} \frac{1}{3n^2 + 4} \]. Determine if it converges or diverges

Solution

Since \( \frac{1}{3n^2 + 4} \leq \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges
our series converges as well.

\[ \sum_{n=1}^{\infty} \frac{2n+5}{3n^2+4} \]. Determine if it converges or diverges

Solution

\( a_n \) looks like \( b_n = \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges

\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2n+5}{3n^2+4} \cdot \frac{n^2}{1} = \lim_{n \to \infty} \frac{2n^3+5n^2}{3n^3+4} = \frac{2}{3} = \text{const} > 0 \]

Our series converges as well.

\[ \sum_{n=1}^{\infty} \frac{2^n(n!)^2}{(2n)!} \]. Determine if it converges or diverges

Solution

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}(n+1)!^2}{(2n+2)!} \cdot \frac{(2n)!}{2^n(n!)^2} \]

\[ = \lim_{n \to \infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{2} < 1 \]

Our series converges.

Integral Test

\[ f(x) = \frac{1}{x \ln(x)^2} \] must be positive, continuous and decreasing

Usually

- for p-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \to \int_{1}^{\infty} \frac{1}{x^p} \, dx \)
- if you see \( \ln n \)
- if you recognize a known integral, e.g. \( \sum_{n=1}^{\infty} \frac{n}{n^e} \to \int_{1}^{\infty} xe^{-x} \, dx \)

Comparison Test

[rarely used! they usually mention it in the question]

larger converges \( \Rightarrow \) smaller too
smaller diverges \( \Rightarrow \) larger too

Limit Comparison Test

[Popular!]
We usually compare with a p-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) conv if \( p > 1 \) div if \( p \leq 1 \)
\( \leftarrow \) any positive constant will do
[we know the result beforehand]

Ratio Test

[Popular!]
Use if you see \( n! \), \( a^n \) or similar expressions e.g. \( 1 \times 3 \times \cdots \times (2n+1) \)
\( \leftarrow < 1, \) converges
\( > 1, \) diverges
\( = 1, \) don't know (use another test)
## 10. Alternating Series

### Alternating Series Test

For \( \sum (-1)^n a_n \)

- \( \lim_{n \to \infty} a_n = 0 \)
- The sequence \( a_n \) decreases

### Solution

The series converges

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+4}
\]

### Approximation

\[ S_0 = 0.142, \quad S_2 = 0.120, \quad S_4 = 0.110 \]
\[ S_1 = 0.042, \quad S_3 = 0.057, \quad S_5 = 0.064 \]

The result is always between two consecutive values so it is above 0.064, and thus above 0.06

### Error of approximation

When you accept \( S_n \) as an approximation then

\[ |\text{error}| < a_{n+1} \]

### Absolute convergence vs Conditional convergence

If the series of the absolute values converges the original also converges (absolutely)

If the series of the absolute values diverges, check original by the alternating series test (it may converge conditionally)

### Solution

The sequence \( \frac{1}{3n+4} \) decreases (see the graph in GDC)

### Error of approximation

\[ S_0 = 0.142, \quad S_2 = 0.120, \quad S_4 = 0.110 \]
\[ S_1 = 0.042, \quad S_3 = 0.057, \quad S_5 = 0.064 \]

The result is always between two consecutive values so it is above 0.064, and thus above 0.06

### Error of approximation

When you accept \( S_n \) as an approximation then

\[ |\text{error}| < a_{n+1} \]

### Absolute convergence vs Conditional convergence

If the series of the absolute values converges the original also converges (absolutely)

If the series of the absolute values diverges, check original by the alternating series test (it may converge conditionally)

### Solution

The series converges

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+4}
\]
11. Power Series

### Solution

\[
\sum_{n=1}^{\infty} \frac{x^n}{3n^3}, \text{ Find radius and interval of convergence}
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)3^{n+1}} \cdot \frac{3^n}{x^n} \right| = \lim_{n \to \infty} \frac{n}{3n+3} |x| = \frac{|x|}{3}
\]

We solve the equation \( R = \frac{1}{3} \)

So the radius of convergence is \( R = 3 \)

For \( x = 3 \) the series becomes \( \sum_{n=1}^{\infty} \frac{1}{n} \), so it diverges

For \( x = -3 \) the series becomes \( \sum_{n=1}^{\infty} \left( -\frac{1}{n} \right) \), so it converges

Hence, the interval of convergence is \( x \in [-3, 3] \)

### Always Ratio Test

[the result can be something like that i.e. \( a|x| \) or \( O \), or \( \infty \); see details in the last box]

\( \approx \) This means that the series converges for \( x \in ]-3, 3[ \)

\( \approx \) diverges for \( x \) outside we don't know yet for \( x = \pm 3 \)

\( \approx \) Check the endpoints

\[
\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 3^n}, \text{ Find radius & interval of convergence}
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-5)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{3^n}{(x-5)^n} \right| = \lim_{n \to \infty} \frac{n}{3n+3} |x-5| = \frac{|x-5|}{3}
\]

We solve the equation \( R = \frac{1}{3} \)

So the radius of convergence is \( R = 3 \)

For \( x = 8 \) the series becomes \( \sum_{n=1}^{\infty} \frac{1}{n} \), so it diverges

For \( x = 2 \) the series becomes \( \sum_{n=1}^{\infty} (-\frac{1}{n}) \), so it converges

Hence, the interval of convergence is \( x \in [2, 8] \)

### Always Ratio Test

Exactly the same procedure but center of convergence = 5

Interval = \( ]5-R, 5+R[ \)

(and check the endpoints)

\( \approx \) This means that the series converges for \( x \in ]2, 8[ \)

\( \approx \) diverges for \( x \) outside we don't know yet for \( x = 2, 8 \)

\( \approx \) Check the endpoints

| Ratio test \( a|x| \) | Radius | Interval of convergence |
|------------------|--------|------------------------|
| \( \frac{1}{a} \) | \( R = \frac{1}{a} \) | \( x \in ]-R, R[ \) and check at \( x = \pm R \) |
| \( O \) | \( R = \infty \) | \( x \in ]-\infty, \infty[ \) |
| \( \infty \) | \( R = 0 \) | \( x = 0 \) |

The limit \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) results in \( a|x| \) or \( O \) or \( \infty \)
## 12. Maclaurin Series – Taylor Series

### Find the Maclaurin series for \( f(x) = \ln\left(\frac{1}{1-x}\right) \), up to \( x^3 \)

#### Solution

1. \( f(x) = -\ln(1-x) \) \( \Rightarrow f(0) = 0 \)
2. \( f'(x) = \frac{1}{1-x} \) \( \Rightarrow f'(0) = 1 \)
3. \( f''(x) = \frac{1}{(1-x)^2} \) \( \Rightarrow f''(0) = 1 \)
4. \( f'''(x) = \frac{2}{(1-x)^3} \) \( \Rightarrow f'''(0) = 2 \)

Thus,

\[
f(x) = f(0) + f'(0) \frac{x}{1!} + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + \ldots
\]

\[
\Rightarrow f(x) = 0 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots
\]

### Lagrange form of error

#### Approximation

\[ f(x) = f(0) + f'(0) \frac{x}{1!} + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + \ldots \]

\[ \Rightarrow f(x) = 0 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \]

#### Max value for \( c = 0.5 \)

\[ |R_3(0.5)| = \left| \frac{f^{(4)}(c)}{4!} \frac{0.5^4}{0.5^4} \right| \leq \frac{1}{4} \frac{0.5^4}{0.5^4} = \frac{1}{4} = 0.25 \]

The actual value of \( f(0.5) \) is \( \ln(2) = 0.693 \)

So the actual error is \( \ln(2) - 0.666 = 0.026 \) (by GDC)

### Find the approximation for \( f(0.5) \) given by the Maclaurin series above. Estimate the error using the Lagrange form. Compare with the actual error.

#### Solution

- \( f(0.5) = 0.5^2 + 0.5^3 = 0.666 \)
- \( R_3(0.5) = \frac{f^{(4)}(c)}{4!} \frac{0.5^4}{0.5^4} \) where \( 0 < c < 0.5 \)

We need \( f^{(4)}(x) = \frac{6}{(1-x)^4} \). Thus

\[
|R_3(0.5)| = \left| \frac{6}{24(1-c)^4} \frac{0.5^4}{0.5^4} \right| \leq \frac{1}{4} \frac{0.5^4}{0.5^4} = \frac{1}{4} = 0.25
\]

The actual value of \( f(0.5) \) is \( \ln(2) = 0.693 \)

So the actual error is \( \ln(2) - 0.666 = 0.026 \) (by GDC)

### Taylor series for \( f(x) = \ln\left(\frac{1}{1-x}\right) \), up to \( (x-a)^3 \), \( a=0.5 \)

#### Solution

- \( f(0.5) = \ln(2) \), \( f'(0.5) = 2 \), \( f''(0.5) = 4 \), \( f'''(0.5) = 16 \)

Thus,

\[
f(x) = f(0.5) + f'(0.5)(x-0.5) + f''(0.5)\frac{(x-0.5)^2}{2!} + \ldots
\]

\[
\Rightarrow f(x) = \ln(2) + 2(x-0.5) + 2(x-0.5)^2 + \frac{8}{3}(x-0.5)^3 + \ldots
\]

Just use the formula in the booklet!

We find (as above) the first three derivatives and the values at \( x=0.5 \)

Notice: the result is a power series, namely \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) with \( R=1 \), interval of conv \( x \in [-1,1] \)

Now we know that this series converges to \( f(x) = \ln\left(\frac{1}{1-x}\right) \)

We need \( f^{(4)}(x) = \frac{6}{(1-x)^4} \). Thus

\[
|R_3(0.5)| = \left| \frac{6}{24(1-c)^4} \frac{0.5^4}{0.5^4} \right| \leq \frac{1}{4} \frac{0.5^4}{0.5^4} = \frac{1}{4} = 0.25
\]

The actual value of \( f(0.5) \) is \( \ln(2) = 0.693 \)

So the actual error is \( \ln(2) - 0.666 = 0.026 \) (by GDC)
Let \( \frac{dy}{dx} = x^2 + 5xy + 4y^2 \), and \( y = 1 \) when \( x = 0 \).

Find the Maclaurin series of \( y = f(x) \) up to \( x^2 \).

**Solution**

We know already \( f(0) = 0 \) and \( f'(0) = \left. \frac{dy}{dx} \right|_{x=0} = 4 \).

We need \( \frac{d^2y}{dx^2} = 2x + 5y + 5x \frac{dy}{dx} + 8y \frac{dy}{dx} \).

Thus

\[
\begin{align*}
  f''(0) &= \left. \frac{d^2y}{dx^2} \right|_{x=0} = 0 + 5 + 32 = 37 \\
  f(x) &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots = 1 + 4x + \frac{37}{2} x^2 + \cdots
\end{align*}
\]

Find the Maclaurin series for \( f(x) = e^{-x^2} \), up to \( x^6 \).

**Solution**

We know that

\[
 e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

Thus

\[
 e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots
\]

Based on the maclaurin series of \( \sin x \) find the Maclaurin series for \( \cos x \) by using term by term

(a) Differentiation

(b) Integration

**Solution**

\[
\begin{align*}
  \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
  (a) \quad \cos x &= (\sin x)' = \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
  (b) \quad \int_0^x \sin x \, dx &= \int_0^x \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \, dx = \left[ \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \right]_0^x \\
  \Rightarrow -\cos x + 1 &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\
  \Rightarrow \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots
\end{align*}
\]