FURTHER MATH HL
TEST IN
MATRICES – VECTOR SPACES

Date: 23 October 2018
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Section A: Without GDC

Name:____

Marks: /50

Questions

1. [Maximum mark: 6]

Consider the matrix $A = \begin{pmatrix} k & 1 & 1 \\ k & 2 & k-1 \\ k & 0 & k-2 \end{pmatrix}$

(a) Find the value of k for which $A^T - A = O$. [2]

(b) Find all possible values of k for which the matrix is **not** invertible. [4]

2. [Maximum mark: 8]

Matrices A, B and C are defined as $\boldsymbol{A}=\begin{pmatrix}1 & 5 & 1\\ 3 & -1 & 3\\ -9 & 3 & 7\end{pmatrix}, \boldsymbol{B}=\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 0\\ 0 & 3 & 1\end{pmatrix}, \boldsymbol{C}=\begin{pmatrix}8\\ 0\\ -4\end{pmatrix}$.

(a) Given that AB = $\begin{pmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{pmatrix}$, find a. [1]

(b) Hence, or otherwise, find A^{-1}. [2]

(c) Find the matrix X, such that $AX = C$. [3]

(d) Find the matrix X, such that $XA^{-1} = C^{T}$. [2]

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3. [Maximum mark: 6]

Show that the vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that satisfy the equation

$$2x + 3y + 5z = 0$$

form a subspace of R^3 of dimension 2.

4. [Maximum mark: 12]

(a) Show that for any matrix A such that $A^3 = \mathbf{O}$ the inverse of $\boldsymbol{I} - A$ is $\boldsymbol{I} + A + A^2$ [3]

Let $A = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$

(b) Find A^2 and A^3 [2]

(c) **Hence** find the inverse of $\begin{pmatrix} 1 & -x & -y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix}$ [3]

(d) Find the 3x3 matrix X such that $X = A + AX$ [4]

5. [Maximum mark: 12]

Consider the system of linear equations

$$\begin{aligned} x + 2y + z &= 3 \\ -x + 2y + 3z &= 1 \\ -2x + y + (a+2)z &= k \end{aligned}$$

where $a \in \mathbb{R}$.

(a) Given that $a = 0$,

(i) show that the system has a unique solution

(ii) find the unique solution in terms of k. [5]

(b) Find the values of a and k for which the system is inconsistent. [3]

(c) Given that the system has infinitely many solutions,

(i) find the values of a and k

(ii) find general solution of the system. [4]

6. [Maximum mark: 6]

Let u and v are linearly independent vectors of R^n.

(a) Show that $2u + 3v$ is a linearly independent vector. [2]

(b) For a vector $w \in R^n$, prove that

u, v, w are linearly dependent **if and only if** w is a linear combination of u and v. [4]

Section B: With GDC

Name:____

Marks: /50

Questions

7. [Maximum mark: 5]

(a) Define the terms **null space** and **nullity** for a $m \times n$ matrix A. [2]

(b) Show that

the nullity of A is equal to the nullity of A^T **if and only if** A is a square matrix. [3]

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8. [Maximum mark: 5]

Let $u = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$.

(a) Show that u and v are linearly independent, by using the definition. [3]

(b) Express $w = \begin{pmatrix} 7 \\ 0 \\ 28 \end{pmatrix}$ as a linear combination of u and v. [2]

9. [Maximum mark: 5]

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(a) Find $\det(A - I)$ [1]

(b) Solve the equation $XA - A = X$ [4]

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10. [Maximum mark: 10]

It is given that $V_1 = \{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} | x, y \in R \}$ is a subspace of R^3.

(a) Show that $V_2 = \{ \begin{pmatrix} a+2b \\ a \\ b \end{pmatrix} | a, b \in R \}$ is also a subspace of R^3. [4]

(b) Find a basis and hence the dimension of V_2. [2]

(c) Describe the subspace $V_1 \cap V_2$ and find its dimension. [2]

(d) Show that $V_1 \cup V_2$ is **not** a subspace of R^3. [2]

11. [Maximum mark: 10]

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 0 \end{pmatrix}$$

(a) Explain why the columns of A are **linearly dependent**. [1]

(b) Find the rank of A. [2]

(c) Deduce a conclusion for the rows of A. [1]

(d) Find the **column space** of A in the simplest form. [1]

(e) Find all vectors $X \in R^4$ such that $AX = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ [3]

(f) Find all row vectors Y such that $YA^T = \begin{pmatrix} 2 & 2 & 3 \end{pmatrix}$ [2]

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12. [Maximum mark: 15]

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Given that $X = B - A^{-1}$ and $Y = B^{-1} - A$

(i) find X and Y;

(ii) does $X^{-1} + Y^{-1}$ have an inverse? Justify your conclusion. [4]

(b) Prove by induction that $A^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$. [6]

(c) Given that $\left(A^n\right)^{-1} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$,

(i) find x and y in terms of n,

(ii) hence find an expression for $A^n + \left(A^n\right)^{-1}$ [5]

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