HAEF IB - MATH HL
TEST 6
STATISTICS AND PROBABILITY

Date: 19 October 2018
by Christos Nikolaidis

Paper 1: Without GDC

Name: SOLUTIONS

Marks: /40

Questions

1. [Maximum mark: 4]

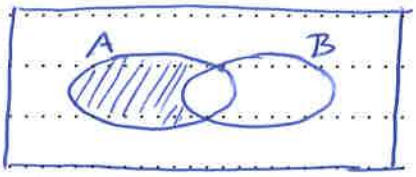
(a) Given that $P(A' \cup B) = 0.53$ find $P(A \cap B')$. [2]

(b) Show that mutually exclusive, non-empty events are **not** independent. [2]

(a) $A' \cup B$ and $A \cap B'$ are complementary

$(A' \cup B)' = A'' \cap B' = A \cap B'$

or by Venn Diagram:



Hence $P(A \cap B') = \boxed{0.47}$

(b) $P(A \cap B) = 0$ [mutually exclusive]

$P(A)P(B) \neq 0$ [not empty]

Hence $P(A \cap B) \neq P(A)P(B)$

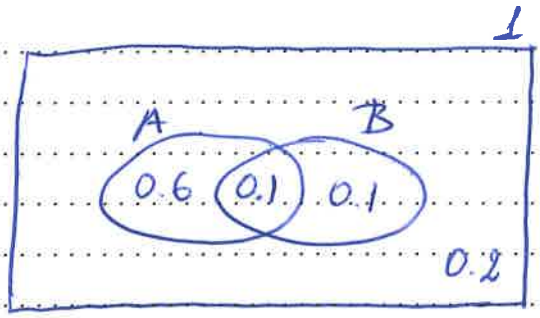
2. [Maximum mark: 6]

Given that $P(A) = 0.7$, $P(B) = 0.2$ and $P(A' \cup B') = 0.9$, find

(a) $P(A \cup B)$ [3]

(b) $P(A' \mid A \cup B)$ [3]

(a)

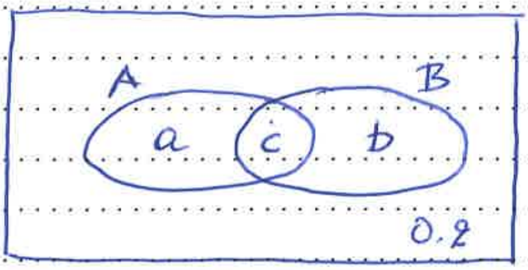


$P(A \cup B) = \boxed{0.8}$

(b) $P(A' \mid A \cup B) = \dfrac{0.1}{0.8} = \boxed{\dfrac{1}{8}}$

3. [Maximum mark: 6]

Let $P(A \mid B) = 0.6$ and $P(B \mid A) = 0.5$. Given that $P(A' \cap B') = 0.2$, find $P(B' \mid A')$



$P(A|B) = 0.6 \Rightarrow \dfrac{c}{b+c} = 0.6 \Rightarrow c = 0.6b + 0.6c$

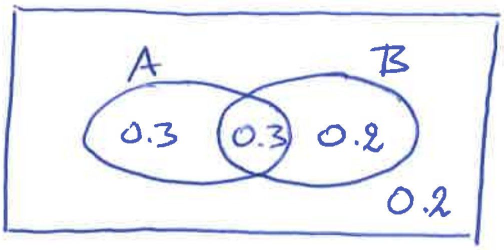
$\Rightarrow 0.4c = 0.6b \Rightarrow b = \frac{2}{3}c$

$P(B|A) = 0.5 \Rightarrow \dfrac{c}{a+c} = 0.5 \Rightarrow c = 0.5a + 0.5c$

$\Rightarrow 0.5c = 0.5a \Rightarrow a = c$

$a + b + c = 0.8 \Rightarrow c + \frac{2}{3}c + c = 0.8 \Rightarrow \frac{8}{3}c = 0.8 \Rightarrow c = 0.3$

Hence $a = 0.3$ $b = 0.2$



$P(B'|A') = \dfrac{0.2}{0.4} = \boxed{0.5}$

(in fact A, B are independent)

4. [Maximum mark: 14]

The probability density function of a continuous random variable X is given by

$$f(x)=\frac{k}{x^2+4}, \quad \text{for } x\in[0,2]$$

(a) Find the value of k. [3]

(b) Show that the function f is decreasing in the domain $x\in[0,2]$. [2]

(c) Find the mode of X. [1]

(d) Find $E(X)$ in the form $\frac{\ln a}{\pi}$ [4]

(e) Find $E(X^2)$ [4]

(a) $\int_0^2 \frac{k}{x^2+4}\,dx=1 \Rightarrow \frac{k}{2}\left[\arctan\frac{x}{2}\right]_0^2=1$

$\Rightarrow \frac{k}{2}\arctan 1 = 1 \Rightarrow \frac{k\pi}{8}=1 \Rightarrow \boxed{k=\frac{8}{\pi}}$

(b) $f'(x) = -\frac{2kx}{x^2+4} < 0$ for $x\in[0,2]$

Hence f is decreasing

(c) $\text{MODE} = x_{max} = \boxed{0}$

(d) $E(X)=\int_0^2 \frac{kx}{x^2+4}dx=\frac{8}{\pi}\int_0^2\frac{x}{x^2+4}dx=\frac{4}{\pi}\left[\ln(x^2+4)\right]_0^2$

$=\frac{4}{\pi}(\ln 8-\ln 4)=\frac{4}{\pi}\ln\frac{8}{4}=\frac{4}{\pi}\ln 2$

$=\boxed{\frac{\ln 16}{\pi}}$

$$E(X^2) = \int_0^2 \frac{kx^2}{x^2+4}\,dx = \frac{8}{\pi}\int_0^2 \frac{x^2+4-4}{x^2+4}\,dx$$

$$= \frac{8}{\pi}\int_0^2 \left(1 - \frac{4}{x^2+4}\right)dx$$

$$= \frac{8}{\pi}\left[x - 2\arctan\frac{x}{2}\right]_0^2$$

$$= \frac{8}{\pi}\left(2 - 2\arctan 1\right)$$

$$= \frac{8}{\pi}\left(2 - \frac{\pi}{2}\right) = \boxed{\frac{16}{\pi} - 4}$$

5. [Maximum mark: 10]

The probability distribution of a discrete random variable X is given by

$$P(X=x)=\frac{k}{x^2+2}, \quad \text{for } x=0,1,2$$

(a) Find the value of k [3]

(b) Find $Var(X)$ [5]

(c) Find the mode of X [1]

(d) Find the median of X [1]

X	0	1	2
$P(X=x)$	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{6}$

(a) $\frac{k}{2}+\frac{k}{3}+\frac{k}{6}=1 \Rightarrow 3k+2k+k=6 \Rightarrow 6k=6 \Rightarrow \boxed{k=1}$

(b) $E(X)=0\cdot\frac{1}{2}+1\cdot\frac{1}{3}+2\cdot\frac{1}{6}=\frac{2}{3}$

$E(X^2)=0^2\frac{1}{2}+1^2\frac{1}{3}+2^2\frac{1}{6}=1$

$Var(X)=1-\left(\frac{2}{3}\right)^2=1-\frac{4}{9}=\boxed{\frac{5}{9}}$

(c) $\boxed{\text{MODE}=0}$

(d) $\boxed{\text{MEDIAN}=0.5}$

Paper 2: With GDC

Name:____

Marks:	/40

Questions

6. [Maximum mark: 6]

Consider the following grouped data

x	frequency	cumulative frequency
[0,10[	15	15
[10,20[	15	30
[20,30[	10	40
[30,40[	30	70
[40,50]	10	80

Find

(a) The number of values of x in the interval [10,30[[2]

(b) The mean [1]

(c) The variance [2]

(d) The median [1]

(a) 25

(b) $\bar{x} = 25.625$

(c) $\sigma^2 = 13.45^2 \cong 181$

(d) 30

7. [Maximum mark: 8]

The probability density function of a continuous random variable X is given by

$$f(x) = \frac{k}{x^3 + 4}, \quad \text{for } x \in [0,2]$$

Find the following, correct to 3 s.f.

(a) the value of k [2]

(b) the values of $E(X)$ and $Var(X)$ [4]

(c) the median of X [2]

(a) $\int_0^2 \frac{k}{x^3+4}\,dx = 1 \Rightarrow k\int_0^2 \frac{1}{x^3+4}\,dx = 1$

$\Rightarrow 0.3741k = 1 \Rightarrow k = 2.67308 \doteq \boxed{2.67}$

(b) $E(X) = \int_0^2 \frac{2.67308x}{x^3+4}\,dx = \boxed{0.831}$

$E(X^2) = \int_0^2 \frac{2.67308x^2}{x^3+4}\,dx = 0.9788928$

$Var(X) = 0.9788928 - 0.8311297^2 = \boxed{0.288}$

(c) $\int_0^M \frac{2.67308}{x^3+4}\,dx = \frac{1}{2} \Rightarrow \boxed{M = 0.769}$ (by GDC SOLVEN)

8. [Maximum mark: 8]

A box contains 10 **red** balls and 10 **black** balls.

Vrasidas selects 3 balls (without replacement).

(a) Find the probability that only red balls are selected. [2]

(b) Find the probability that only one red ball selected. [2]

Melpomeni selects 4 balls (without replacement).

(c) Find the probability that all balls have the same color. [2]

(d) Find the probability that the red balls are more that the black ones. [2]

10 BLACK
10 RED
20

(a) $\frac{10}{20} \cdot \frac{9}{19} \cdot \frac{8}{18} = \boxed{\frac{2}{19}}$ $(\cong 0.105)$

(b) $\frac{10}{20} \cdot \frac{10}{19} \cdot \frac{9}{18} \times 3 = \boxed{\frac{15}{38}}$ $(\cong 0.395)$

(c) $\frac{10}{20} \cdot \frac{9}{19} \cdot \frac{8}{18} \cdot \frac{7}{17} \times 2 = \boxed{\frac{28}{323}}$ $(\cong 0.0867)$

(d) $P(3R\text{-}1B) = \frac{10}{20} \times \frac{9}{19} \times \frac{8}{18} \times \frac{10}{17} \times \binom{4}{3} = \frac{80}{323}$

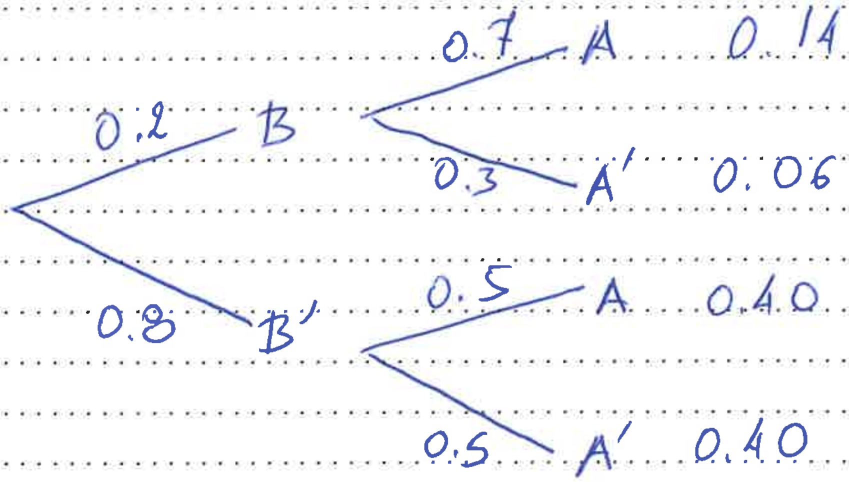
$P(4R) = \frac{14}{323}$ Hence $P = \frac{80+14}{323} = \boxed{\frac{94}{323}}$

OR $P(2R\text{-}2B) = \frac{10}{20} \times \frac{9}{19} \times \frac{10}{18} \times \frac{9}{17} \times \binom{4}{2} = \frac{135}{323}$

Due to symmetry. $P = \frac{1}{2}\left(1 - \frac{135}{323}\right) = \boxed{\frac{94}{323}}$

9. [Maximum mark: 6]

Given that $P(B) = 0.2$, $P(A \mid B) = 0.7$ and $P(A \mid B') = 0.5$ find $P(B' \mid A')$.



$$P(B' / A') = \frac{0.40}{0.06 + 0.40} = \frac{0.40}{0.46} =$$

$$= \boxed{\frac{20}{23}} \quad (= 0.870)$$

10. [Maximum mark: 6]

40 families are reviewed for the number of pets they own. Part of the results is shown below

Number of pets	number of families
0	14
1	a
2	b
3	c
more than 3	none

Given that the mean is 1.1 and the variance is 1.04 complete the table above.

$a+b+c+14=40 \Rightarrow a+b+c=26 \quad (1)$

$\bar{x}=1.1 \Rightarrow a+2b+3c=44 \quad (2)$

$\sum \frac{x^2}{n} = (a+4b+9c)/40$

$\sigma^2 = 1.04 \Rightarrow \frac{a+4b+9c}{40} - 1.1^2 = 1.04$

$\Rightarrow a+4b+9c = 90 \quad (3)$

System $(1), (2), (3)$

$$\boxed{\begin{aligned} a &= 13 \\ b &= 8 \\ c &= 5 \end{aligned}}$$

11. [Maximum mark: 6]

A box contains 1 red ball, 1 green ball and 8 white balls. Players pay a fee of 2€ to participate in the following game. They select two balls

1st prize: RED-GREEN pays n euros back.

2nd prize: RED-WHITE pays 5 euros back.

3rd prize: GREEN-WHITE pays 2 euros back (i.e. the initial fee)

WHITE-WHITE gives nothing.

(a) Find the value of n if the game is fair. [4]

(b) Find the expected loss for the player after ten games if the first prize is $n = 25$ euros. [2]

EITHER

PROFIT	$n-2$	3	0	-2
Prob	$\frac{2}{90}$	$\frac{16}{90}$	$\frac{16}{90}$	$\frac{56}{90}$

$E(\text{Profit}) = 0$

OR

RETURN	n	5	2	0
Prob	$\frac{2}{90}$	$\frac{16}{90}$	$\frac{16}{90}$	$\frac{56}{90}$

$E(\text{Return}) = 2$

(a) $E(\text{Profit}) = 0 \Rightarrow \dfrac{2(n-2) + 3\times16 - 2\times56}{90} = 0$

$\Rightarrow 2n - 4 + 48 - 112 = 0 \Rightarrow 2n = 68$

$\Rightarrow \boxed{n = 34}$

(b) If $n = 25$

$E(\text{Profit}) = \dfrac{2\times23 + 3\times16 - 2\times56}{90} = -\dfrac{18}{90} = -\dfrac{1}{5}$

After ten games $-\frac{1}{5}\times10 = \boxed{-2\,€}$