MATH HL

OPTION

REVISION - SOLUTIONS

SETS, RELATIONS AND GROUPS

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PART B: GROUPS

GROUPS

1. (a) $(a * b) * c = \left(\frac{ab}{a+b}\right) * c = \dfrac{\frac{abc}{a+b}}{\frac{ab}{a+b}+c} = \dfrac{abc}{ab+ac+bc}$ (M1)A1A1

$a * (b * c) = a * \left(\frac{bc}{b+c}\right) = \dfrac{\frac{abc}{a+b}}{a+\frac{bc}{b+c}} = \dfrac{abc}{ab+ac+bc}$ (M1)A1A1

$\therefore (a * b) * c = a * (b * c)$ R1

so $*$ is associative. AG 7

(b) Suppose e is an identity element, then $e * a = a * e = a$ (M1)

$\frac{ea}{e+a} = a$ A1

$ea = ea + a$ M1

ea cancels on both sides so there is no solution for e. R1

i.e. no identity element AG 4

[11]

2. (a) $a \# b = a + b + 1$

Now $b \# a = b + a + 1$ (M1)

Since + is commutative $a \# b = b \# a$ (A1)

$\Rightarrow$ # is also a commutative operation. (AG)

$(a \# b)\#c = (a + b + 1)\#c$

$= a + b + 1 + c + 1$

$= a + b + c + 2$ (A1)

$a\#(b \# c) = a\#(b + c + 1)$

$= a + b + c + 1 + 1$

$= a + b + c + 2$ (A1)

$\Rightarrow$ # is also associative operation. (AG) 4

(b) To show $(\mathbb{R}, \#)$ is a group we need to show closure, identity element exists, inverses exist and it is associative (already shown).

It is closed since $a + b + 1 \in \mathbb{R}$ for $a, b \in \mathbb{R}$. (A1)

There is a unique element $e(e \in \mathbb{R})$ such that

$p \# e = e \# p = p$ where $p \in \mathbb{R} \Rightarrow p + e + 1 = e + p + 1 = p$

$\Rightarrow e = -1$ as identity element (A1)

There are unique inverse elements for each element in $\mathbb{R}$ such that

$p \# p^{-1} = p^{-1} \# p = -1$ (M1)

$\Rightarrow p + p^{-1} + 1 = p^{-1} + p + 1 = -1$

$\Rightarrow p^{-1} = -p - 2$ (A1)

Hence $(\mathbb{R}, \#)$ forms a group. (AG) 4

[8]

3. (a) $a, b \in T \Rightarrow a * b \in T$ (A1)

if $a*b=1$, $ab-a-b+2=1, \Rightarrow ab-a-b+1=0$ (M1)(A1)

$\Rightarrow (a-1)(b-1)=0 \Rightarrow a=1$, or $b=1$ contradiction (M1)(R1)

so $a*b \in T$, i.e. closed (AG) 5

(b)

$(x*y)*z=(xy-x-y+2)*z$. *(A1)*

$= xyz-xz-yz+2z-xy+x+y-2-z+2$ *(A1)*

$xyz-yz-zx-xy+x+y+z$ *(AG)*

$x*(y*z)=x*(yz-y-z+2)$ *(A1)*

$= xyz-xy-xz+2x-x-yz+y+z-2+2$ *(A1)*

$=(x*y)*z$ *(A1)*

Note: as the operation is clearly commutative, there is no need to check **both** left **and** right identity, or **both** left **and** right inverse below

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(c) $a*e=a \Rightarrow e(a-1)=2(a-1) \Rightarrow e=2$ (since $a \neq 1$) (M1)(A1)

Hence 2 is the identity element for this operation. (A1) 3

(d) $a * a' = 2 \Rightarrow aa' - a - a' + 2 = 2 \Rightarrow a'(a-1) = a \Rightarrow a' = a/(a-1)$ M1A1

Hence $3' = 3/2$ A1 3

(e) (i) The formula is true for $n=1$ since $a=(a-1)^1+1$. (R1)

Assume that it is true for $n=k$, i.e. $\overbrace{a*a*\cdots*a}^{k\ times}=(a-1)^k+1$ (M1)

$\overbrace{a*a*\cdots*a}^{k+1\ times}=\left((a-1)^k+1\right)*a=\left((a-1)^k+1\right)a-\left((a-1)^k+1\right)-a+2$ (M1)

$=(a-1)^k \times a+a-(a-1)^k-1-a+2$ (A1)

$=(a-1)^k(a-1)+1$ (A1)

$=(a-1)^{k+1}+1$

so the formula is proven by mathematical induction. (R1) 6

(ii) We require $a*a*\ldots*a=2$ (M1)

so that $(a-1)^n+1=2$ or $(a-1)^n=1$ (A1)

Apart from $a=2$, the identity, the only solution is $a=0$. (A1)

Since $0*0=2$, the element 0 has order 2. (A1) 4

[26]

4. (a) Since $\forall a \in G$, $e \circ a = a \circ e$ because e is the identity element of the group. (R2)

Then $e \in H$. (AG) 2

(b) Let $x, y \in H$, then $(x \circ y) \circ a = x \circ (y \circ a)$ (by associativity) (R1)

$= x \circ (a \circ y)$ (since $y \in H$) (R1)

$= (x \circ a) \circ y$ (associativity) (R1)

$= (a \circ x) \circ y$ ($x \in H$) (R1)

$= a \circ (x \circ y)$ (associativity) (R1)

Therefore, $(x \circ y) \circ a = a \circ (x \circ y)$

$\Rightarrow (x \circ y) \in H$. (AG) 5

(c) $e \circ a = a \circ e$ identity (R1)

$\Rightarrow (x^{-1} \circ x) \circ a = a \circ (x^{-1} \circ x)$ (R1)

$\Rightarrow x^{-1} \circ (x \circ a) = (a \circ x^{-1}) \circ x$ associativity

$\Rightarrow x^{-1} \circ (a \circ x) = (a \circ x^{-1}) \circ x$ $x \in H$ (R1)

$\Rightarrow (x^{-1} \circ a) \circ x = (a \circ x^{-1}) \circ x$ associativity (R1)

Therefore, $x^{-1} \circ a = a \circ x^{-1}$ cancellation law

and $x^{-1} \in H$ (AG) 4

[11]

FINITE GROUPS – CAYLEY TABLES

5. Closure - yes, because the table contains no other elements. ***(R1)***

Identity - yes, d. ***(R1)***

Inverse - yes, every element has an inverse (or d appears in every row and column). ***(R1)***

Associativity - no because, ***(R1)***

$b\#(c\#e) = b\#a = e$ but $(b\#c)\#e = a\#e = b$ ***(A1)(A1)***

[6]

6. (a) **Note:** Award ***(A3)*** if one error, ***(A2)*** if 2 errors, ***(A1)*** if 3 errors, ***(A0)*** for more

	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

(A4) 4

(b) (i) using inverse elements

$(b\#x)*c*a = d*a$

$\Rightarrow b\#x = a$ ***(A1)***

$\Rightarrow d\#b\#x = d\#a$

$\Rightarrow x = d$ ***(A1)***

(ii) $a*(x\#b)*c*a = b*a$

$\Rightarrow a*(x\#b) = c$ ***(A1)***

$\Rightarrow c*a*(x\#b) = c*c$

$\Rightarrow x\#b = b$ ***(A1)***

$\Rightarrow x\#b\#d = b\#d$

$\Rightarrow x = a$ ***(A1)*** 5

[9]

7. (a) The operation table is thus:

$(*)$	1	3	4	9	10	12
1	1	3	4	9	10	12
3	3	9	12	1	4	10
4	4	12	3	10	1	9
9	9	1	10	3	12	4
10	10	4	1	12	9	3
12	12	10	9	4	3	1

(A4) 4

Note: *Award (A3) if one entry is incorrect, (A2) if two entries are incorrect, (A1) if three are incorrect, (A0) if four or more are incorrect.*

(b) $*$ is associative and commutative (known) (A1)

The set is closed under $*$ (A1)

1 is the identity element (A1)

Every element has an inverse because 1 is on each row (or column). (A1) 4

(c) 1 is of order 1
12 is of order 2 (A1)
3 and 9 are of order 3 (A1)
4 and 10 are of order 6 (A1) 3

Note: *If one answer is wrong, award (A1), if two or more answers are wrong award (A0).*

(d) There are four subgroups:
{1}
{1, 12} (A1)
{1, 3, 9} (A2)
{1, 3, 4, 9, 10, 12} 3

[14]

8.

(a) (i) $3 \otimes 5 = 15$ ***(A1)***

(ii) $3 \otimes 7 = 5$ ***(A1)***

(iii) $9 \otimes 11 = 3$ ***(A1)*** 3

(b) (i) The operation table is

⊗	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**
1	1	3	5	7	9	11	13	15
3	3	9	**15**	**5**	(11)	1	7	13
5	5	**15**	(9)	3	13	7	1	11
7	7	**5**	3	(1)	15	13	11	9
9	9	(11)	13	15	1	**3**	5	7
11	11	1	7	13	**3**	(9)	15	5
13	13	7	1	11	5	15	9	3
15	15	13	11	9	7	5	3	1

(A2)

Note: Award ***(A2)*** if the circled numbers are correct, ***(A1)*** if 3 or 4 are correct, ***(A0)*** otherwise. The bold numbers were found in part (a)

(ii) Closure: The table shows that no new elements are generated. ***(R1)***
Identity: 1 is the identity. ***(R1)***
Inverse: Every row and column has a "1". ***(R1)***
(Associative given).
So (S, ⊗) is a group. ***(AG)*** 5

(c) (i) Elements of order 2 are 7, 9, 15. ***(A2)***

Note: Award ***(A1)*** if one correct element is given.

(ii) Elements of order 4 are 3, 5, 11, 13. ***(M1)(A1)***

Note: If no working shown, award ***(M1)(A0)*** if one correct element is given. 4

(d) Using 3 as the generator, a sub-group of order 4 is $\{1, 3, 9, 11\}$. ***(M1)(A1)***

Note: Another possibility is $\{1, 5, 9, 13\}$. 2

[14]

9. (a) $(3*9)*13 = 13*13 = 1$ (M1)

and $3*(9*13) = 3*5 = 1$ (M1)

hence $(3*9)*13 = 3*(9*13)$ (AG) 2

(b) To show that $(U, *)$ is a group we need to show that:

(1) U is closed under $*$. A table is an easy way of showing closure for this finite set.

(*)	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

(C4)

Note: *Award (C4) for a completely accurate table, (C3) for 1 or 2 errors, (C2) for 3 or 4 errors, (C1) for 5 or 6 errors, (C0) for 7 or more errors.*

since for each $a, b \in U$, $a * b \in U$, **closure** is shown. (C1)

(2) Since multiplication is **associative**, it is true in this case too. (C1)

(3) Since $1*a = a*1 = a$ **for all** $a \in U$, 1 is the **identity**. (C2)

(4) 1 appears in each row of the table once, so every element has a unique **inverse**.
$(1^{-1} = 1, 3^{-1} = 5, 5^{-1} = 3, 9^{-1} = 11, 11^{-1} = 9, 13^{-1} + 13)$ (C2) 11

(c) (i) If G is a group and if there exists $a \in G$, such that
$G = \{a^n : n \in \mathbb{Z}\}$
Then G is a cyclic group and a is called a generator. (C2) 2

(ii) By inspection:

3 is a generator since:
$3^2 = 9, 3^3 = 13, 3^4 = 11$ (M1)
$3^5 = 5, 3^6 = 1$ (A1)

Also, 5 is a generator:
$5^2 = 11, 5^3 = 13, 5^4 = 9$ (M1)
$5^5 = 3, 5^6 = 1$ (A1)

9 cannot be a generator since $9^3 = 1$ (C1)

similarly $11^3 = 1$ and $13^2 = 1$. (C1) (C1) 7

(d) Since the order of this group is 6, by Lagrange's Theorem, the proper subgroups can only have orders 2 or 3. (R1)

Since 13 is the only self inverse $13^2 = 1$, (R1)
the only subgroup of order 2 is $\{1, 13\}$ (A1)
No sub-group may include 3 or 5 since these are the generators of the group.
The only elements left are 9 and 11. (R1)
Now, $9*11 = 1$, $9^2 = 11$, and $11^2 = 9$. (M2)
Therefore, $\{1, 9, 11\}$ is the other sub-group. (A1) 7

[29]

PERMUTATION GROUPS

10. (a) Since 3! = 6, order of S = 6. (M1) (R1) 2

(b) Members of S are $p_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$ (AG)

$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ (A2) 2

Note: *Award (A2) for 3 correct permutations;(A1) for 2 (A0) for 1*

$p_3 \circ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, p_4 \circ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ (M1)

$p_3 \circ p_4 \neq p_4 \circ p_3$ (R1) 2

Note: *There are other possibilities to show that the group is not Abelian.*

(c) $p_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_2$

$p_1^3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = p_0.$ (M1)

(Note that p_0 is the identity of the group S.)

Hence $\{p_0, p_1, p_2\}$ form a cyclic group of order 3 under composition. (R1) 2

Note: *Some candidates may write $\{p_0, p_1, p_2\}$ is a subgroup of order 3, (award (A1)), and write the following table, (award (R1)):*

$\circ$	p_0	p_1	p_2
p_0	p_0	p_1	p_2
p_1	p_1	p_2	p_0
p_2	p_2	p_0	p_1

[8]

11. (a) $\begin{pmatrix} a & b & c & d \\ b & d & a & c \end{pmatrix}$ (A1) 1

(b) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}; \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$ (A2) 2

Note: *There are many correct answers for the second permutation.*

(c) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$

$\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}; \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}; \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$ (A1)(A1)(A1)

Let p, q, r, s be the four permutations in the subgroup. Closure is shown by the group table, *i.e.* (M1)

	p	q	r	s
p	p	q	r	s
q	q	r	s	p
r	r	s	p	q
s	s	p	q	r

(A1)

Inverse: each element has an inverse, (M1)

i.e. $p^{-1} = p, q^{-1} = s, r^{-1} = r, s^{-1} = q.$ (A1) 7

Note: *There are other possible answers.*

[10]

GROUPS AND RELATIONS (COSETS)

12. (a) $x^{-1}x = e \in H. \Rightarrow x\,R\,x \Rightarrow R$ is reflexive — M1 R1

$x\,R\,y \Rightarrow x^{-1}y \in H \Rightarrow (x^{-1}y)^{-1} \in H$ — A1

$x^{-1}y(x^{-1}y)^{-1} = e$ so $(x^{-1}y)^{-1} = y^{-1}x$ — A1

$\Rightarrow y^{-1}x \in H \Rightarrow y\,R\,x \Rightarrow R$ is symmetric — R1

$x\,R\,y$ and $y\,R\,z \Rightarrow x^{-1}y \in H$ and $y^{-1}z \in H$

$\therefore (x^{-1}y)(y^{-1}z) \in H$ since H is closed. — A1

$x^{-1}(yy^{-1})z \in H \Rightarrow x^{-1}z \in H$ — A1

$\Rightarrow x\,R\,z \Rightarrow R$ is transitive. — R1

$\therefore R$ is an equivalence relation. — AG 8

(b) $p^3 = q^2 = e \qquad qp = p^2q$

$qp^2 = (qp)p = (p^2q)p$ — A1

$= p^2(qp) = p^2(p^2q) = p^3(pq) = pq$ — A1A1 AG 3

(c) $H = \{e, p^2q\}$

$y\,R\,pq \Rightarrow y^{-1}pq = e \Rightarrow pq = y$ — A1

or $y^{-1}pq = p^2q \Rightarrow pq = yp^2q$

$pq^2 = yp^2q^2\,p = yp^2$ — A1

$p^2 = yp^3$ — A1

$p^2 = y$ — A1

$\therefore$ The equivalence class is $\{p^2, pq\}$ — A1 5

OTHERWISE

The equivalence class of pq is the coset pqH which contains pq and $pqp^2q = ppqq = p^2$.

[16]

Extra question

There are 3 equivalence classes (3 cosets)

$\boldsymbol{H}=\{e, p^2q\}$,

$\boldsymbol{pH}=\{p, q\}$

$\boldsymbol{p^2H} = \{p^2, pq\}$

ISOMORPHISMS

13. (a) f is injective since $f(x) = f(y) \iff 3^x = 3^y \iff x = y$ — (M1)(R1)

f is surjective since if $z \in \mathbb{R}^+$, $x = \log_3(z) \in \mathbb{R}$ and $z = f(x)$ — (M1)(R1)

For every x, y in $(\mathbb{R}, +)$,

$f(x + y) = 3^{(x+y)} = 3^x3^y = f(x) \times f(y)$ — (M1)(A1) 6

(b) $f^{-1}(z) = \log_3(z)$ — (A1) 1

[7]

14. (a) Since $(a + b\sqrt{2})(c + d\sqrt{2}) = ac + 2bd + (ad + bc)\sqrt{2}$,

and $(ac + 2bd)^2 - 2(ad + bc)^2 = (a^2 - 2b^2)(c^2 - 2d^2) \neq 0$,

S is closed under multiplication. — (A2)

$1 = 1 + 0\sqrt{2}$ is the neutral element. — (A1)

Finally, $\dfrac{a - b\sqrt{2}}{a^2 - 2b^2} \in S$ — (M1)

and $\left(\dfrac{a - b\sqrt{2}}{a^2 - 2b^2}\right)(a + b\sqrt{2}) = 1$, so every element of S has an inverse. — (A1) 5

(b) To show that $f(x)$ is an isomorphism, we need to show that it is injective, surjective and that it preserves the operation.

Injection: Let $x_1 = a + b\sqrt{2}$, $x_2 = c + d\sqrt{2}$

$f(x_1) = f(x_2) \Rightarrow a - b\sqrt{2} = c - d\sqrt{2} \Rightarrow (a - c) + (d - b)\sqrt{2} = 0$ (M1)

$\Rightarrow a = c$, and $b = d \Rightarrow x_1 = x_2$ (A1)

Surjection: For every $y = a - b\sqrt{2}$ there is $x = a + b\sqrt{2}$ (M1)(A1)

Preserves operation:

$f(x_1x_2) = f((a + b\sqrt{2})(c + d\sqrt{2})) = f(ac + 2bd +)ad + bc)\sqrt{2})$ (M1)

$= ac + 2bd - (ad + bc)\sqrt{2} = (a - b\sqrt{2})(c - d\sqrt{2})$ (M1)

$(f(a + b\sqrt{2}))(f(c + d\sqrt{2})) = (f(x_1))(f(x_2))$ 6

[11]

15. (a)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(A1) 1

(b)

(*)	*a*	*b*	*c*	*d*
a	*b*	*a*	*d*	*c*
b	*a*	*b*	*c*	*d*
c	*d*	*c*	*a*	*b*
d	*c*	*d*	*b*	*a*

(A4) 4

Notes: *There are many other correct solutions, with a different ordering*
Award (A4) if all entries are correct,(A3) if all but 1 entry are correct, (A2) if all but 2 entries are correct,(A1) if all but 3 entries are correct.

[5]

16. (a)

o	*f*	*g*	*h*	*j*
f	*f*	*g*	*h*	*j*
g	*g*	*f*	*j*	*h*
h	*h*	*j*	*f*	*g*
j	*j*	*h*	*g*	*f*

(A3) 3

Note: *Award (A3) for all correct, (A2) for 1 error, (A1) for 2 errors, (A0) otherwise.*

(b)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

x_5	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

To investigate isomorphisms we can consider the order of elements (M1)
for $+_4$, the identity is 0, 1 has order 4, 2 has order 2 and 3 has order 4, (A1)
for x_5, the identity is 1, 2 has order 4, 3 has order 4 and 4 has order 2, (A1)
for $\circ$, the identity is *f*, and *g*, *h* and *j* all have order 2. (A1)
Hence $+_4$ is isomorphic with x_5. (A1)
Corresponding elements are
$0 \leftrightarrow 1$, $1 \leftrightarrow 2$, $2 \leftrightarrow 4$, $3 \leftrightarrow 3$, OR $0 \leftrightarrow 1$, $1 \leftrightarrow 3$, $2 \leftrightarrow 4$, $3 \leftrightarrow 2$. (A1) 6

Note: *Corresponding elements* ***must*** *be correct for final (A1).*

[9]

17. (a) By using the composition of functions we form the Cayley table

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

(A3)

Note: *For each error in the above table deduct one mark up to a maximum of three marks.*

From the table, we see that $(T, \circ)$ is a closed and is commutative. (R1)
f_1 is the identity. (A1)
$f_i^{-1} = f_i$, $i = 1, 2, 3, 4$. (A1)

Since the composition of functions is an associative binary operation an Abelian group. (AG) 6

(b) The Cayley table for the group $(G, \lozenge)$ is given below:

$\lozenge$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

(A2)

Note: *For each error in the entries deduct one mark up to a maximum of two marks.*

Define $f : T \mapsto G$ such that $f(f_1) = 1, f(f_2) = 3, f(f_3) = 5$ and $f(f_4) = 7$ (M1)
Since distinct elements are mapped onto distinct images, it is a bijection.(R1)
Since the two Cayley tables match, the bijection is an isomorphism. (R1)
Hence the two groups are isomorphic. (AG) 5

[11]

18. (a) B is the set $\{1, i, -1, -i\}$ (A1)
This set is closed under multiplication.
Associative, since it is normal complex number multiplication. (R1)
The identity element is 1. (R1)
The inverse of i is –i, and vice versa, 1 and –1 are self inverses. (R1)

(b)

×	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

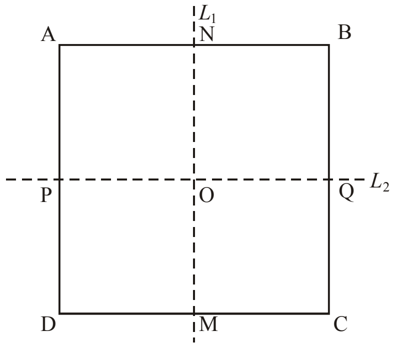
(A2)

(c) Order of 1 is 1
Order of 3 is 4, since $3^4 = 1$ (A1)
Order of 7 is 4, since $7^4 = 1$ (A1)
Order of 9 is 2, since $9^2 = 1$ (A1)

(d) The two groups will have a bijection in which the following correspond:
$1 \leftrightarrow 1$, $3 \leftrightarrow i$, $7 \leftrightarrow i$, and $9 \leftrightarrow -1$ (or $3 \leftrightarrow -i$, $7 \leftrightarrow i$) (A1)
Both groups have the same structure, the bijection preserves the operation. (R1)

[11]

19.



(a)

$\circ$	U	H	V	K		
U	U	H	V	K		
H	H	U	K	V		
V	V	K	U	H		
K	K	V	H	U	(A4)	4

Note:*(A4) for 15-16 correct entries,(A3) for 13-14, (A2) for 11-12, (A1) for 9-10, (A0) o/w*

(b) Closure: U, H, K and V are the only entries in the table. So it is closed. (A1)
Identity: U, since $UT = TU = T$ for all T in S. (A1)
Inverses: $U^{-1} = U, H^{-1} = H, V^{-1} = V, K^{-1} = K$ (A1)
Associativity: Given (AG)
Hence $(S, \circ)$ forms a group. (R1) 4

(c) $C = \{1, -1, \mathrm{i}, -\mathrm{i}\}$

$\Diamond$	1	–1	i	–i		
1	1	–1	i	–i		
–1	–1	1	–i	i		
i	i	–i	–1	1		
–i	–i	i	1	–1	(A3)	3

Note: *Award (A3) for 15-16 correct entries, (A2) for 13-14, (A1) for 11-12, (A0) for fewer*

(d) Suppose $f : S \to C$ is an isomorphism.
Then $f(U) = 1$, the identity in C, since f preserves the group operation.(M1)(C1)
Assume $f(H) = \mathrm{i}$, $1 = f(U) = f(H \circ H) = f(H) \Diamond f(H)$. (A1)
But $f(H) = \mathrm{i}$, and i is not its own inverse, so f is not an isomorphism. (R1) 4

Note: *Accept other correctly justified solutions.*

[15]

THEORETICAL

20. Let $a^{-1} = b$ (M1)
Then $e = b \times a = b \times a \times a$ (M1)
so that $e = (b \times a) \times a = e \times a$ (M1)
and therefore $e = a$ (M1)(AG)

***Note**: There are other correct solutions.*

[4]

21. (a) If G is a group and H is a subgroup of G then the order of H is a divisor of the order of G. (A2) 2

(b) Since the order of G is 24, the order of a must be 1, 2, 3, 4, 6, 8, 12 or 24 (R2)
The order cannot be 1, 2, 3, 6 or 12 since $a^{12} \neq e$ (R1)
Also $a^8 \neq e$ so that the order of a must be 24 (R1)
Therefore, a is a generator of G, which must therefore be cyclic. (R1) 5

[7]

22. (a) A cyclic group is a group which is generated by one of its elements (or words to that effect). (M2) 2

(b) We can assume that (G, #) has at least two elements and hence contains an element, say b, which is different from e, its identity. (R1)
The order of b is equal to the order q of the subgroup it generates. (M1)
By Lagrange's theorem q must be a factor of p and since p is prime either $q = 1$ or $q = p$. (R1)
Since $b \neq e$ we see that $q \neq 1$ and therefore $q = p$. (R1)
But if the order of b is p then b generates (G, #) which is therefore cyclic. (R1) 5

[7]

23.

For $a \in H$, $a^{-1} * a = e \in H$ so H contains the identity. ***(A1)***
For $a \in H$, $a^{-1} * e = a^{-1} \in H$ so H contains all the inverse elements. ***(A1)***
$*$ is associative on G and therefore on H. ***(A1)***
For $a, b \in H, a^{-1} \in H$ so $(a^{-1})^{-1} * b = a * b \in H$ so closure confirmed. ***(A1)(A1)***
The four requirements are satisfied so (H, *) is a subgroup. ***(R1)***

[6 marks]

24. Consider $a * b$. This cannot be a or b since $a * b = a \Rightarrow b = e$ which is not the (M1)
case and similarly for b. So $a * b =$ either e or c. (R1)
If $a * b = e$, then a,b form an inverse pair so $b * a = e$. (R1)
Suppose $a * b = c$. Consider $b * a$. As before, this cannot equal a or b and it cannot equal e either because that would imply that $a * b = e$ which it is not. (R1)
It follows that $b * a = c$. (R1)
Thus in both cases, $a * b = b * a$. (R1)

[6]

25. Given (G, *) is a cyclic group with identity e and $G \neq \{e\}$ and G has no proper subgroups.
If G is of composite finite order and is cyclic, then there is $x \in G$ such that x generates G. (R1)
If $|G| = p \times q$, $p, q \neq 1$, then $\langle x^p \rangle$ is a subgroup of G of order q which is (M1)
impossible since G has no non-trivial proper subgroup. (R1)
Suppose the order of G is infinite. Then $\langle x^2 \rangle$ is a proper subgroup of G which (M1)
contradicts the fact that G has no proper subgroup. (A1)
So G is a finite cyclic group of prime order. (R1)

[6]

26. **If** one of the sets H and K is contained in the other then either $H \cup K = H$ or $H \cup K = K$.
In either case it is a subgroup of $(G, \circ)$. (C2)

Only if:

Conversely, suppose that $(H \cup K, \circ)$ is a subgroup of $(G, \circ)$ and that H is not contained in K. (M1)
Then there exists an element b of H which is not included in K. (C1)
Let a be any element of K.
Then $ab \in H \cup K$ (since $(H \cup K, \circ)$ is a group). (C1)
If $ab \in K$ then $b = a^{-1}ab \in K$ which is a contradiction of our hypothesis. (C1)
Hence $ab \notin K$ and therefore $ab \notin H$ so that $abb^{-1} \in$ H (C1)
which shows that $K \subseteq H$ since a was any element of K. (C1)
Therefore $H \subseteq K$ or $K \subseteq H$. (AG)

OR

Proof by contradiction: (M1)
$K \not\subset H$ then there exists $m \in K, m \notin H$ (C1)
And
$H \not\subset K$ then there exists $n \in H, n \notin K$. (C1)
Suppose $m \circ n \in H$ then $m \circ n \circ n^{-1} \in H$ is a contradiction (C1)
Suppose $m \circ n \notin K$ then $n = m^{-1} \circ m \circ n \in K$ is a contradiction (C1)
Hence $m \circ n \notin H \cup K$ a contradiction (C1)
Therefore $H \subseteq K$ or $K \subseteq H$ (AG) 8

[8]

27. (a) Let $(G, \circ)$ and $(H, \bullet)$ be two groups. They are said to be isomorphic if there exists a one-to-one transformation $f: G \to H$ which is surjective (onto) with the (C1)
property that for all $x, y \in G, f(x \circ y) = f(x) \bullet f(y)$. (C1) 2

__Note:__ Some candidates may say that the groups (G, ∘) and (H, •) are isomorphic if they have the same Cayley table (or group table). In that case award (C1).

(b) Since $f: G \to H, f(x) \in H$ for some $x \in G$.
Since e' is the identity element in H, (M1)
$e' \bullet f(x) = f(x) = f(x \circ e) = f(e) \bullet f(x)$. (M1)(A1)
By the right cancellation law, $e' = f(e)$. (R1) 4

(c) Suppose $G = \langle a \rangle$, the cyclic group generated by a, *i.e.* n is the smallest positive integer such that $a^n = e$, the identity in G. (C1)
Let $f: G \to H$ be an isomorphism. Let $f(a) = b \in H$.
$f(a^2) = f(a \circ a) = f(a) \bullet f(a) = (f(a))^2$. (M1)
In general $f(a^m) = (f(a))^m$, $1 \le m \le n$. (A1)
By (iii) (b) $(f(a))^n = e'$, the identity in H. Hence $b^n = e'$ and consequently H is a cyclic group of order n with generator b. (R1) 4

[10]

28. (a) Suppose a is of order n and is a^{-1} of order m.
Therefore $e = e*e = (a^{-1})^m*a^n$ (M1)

If $m > n$, then $e = (a^{-1})^{m-n}*(a^{-1})^n*a^n = (a^{-1})^{m-n}*(a^{-1}*a)^n$. (M1)
Hence $e = (a^{-1})^{m-n}$. This implies a^{-1} is of order $m - n < m$
which is a contradiction. So m is not greater than n. (R1)

If $m < n$, $e = (a^{-1})^m*a^m*a^{n-m} = (a^{-1}*a)^m*a^{n-m}$ (M1)
Hence $e = a^{n-m}$, which implies a is of order $n - m < n$.
This is a contradiction. (R1)
Therefore $m = n$. (AG) 5

(b) Let $S(m)$ be the statement: $b^m = p^{-1}*a^m*p$.
$S(1)$ is true since we are given $b = p^{-1}*a*p$ (A1)
Assume $S(k)$ as the induction hypothesis. (M1)
$b^{k+1} = b^k*b = (p^{-1}*a^k*p)*(p^{-1}*a*p) = p^{-1}*a^{k+1}*p$ (M1)(R1)
which proves $S(k + 1)$.

Hence, by mathematical induction $b^n = p^{-1}*a^n*p$ ($n = 1, 2, \ldots$). (AG) 4

[9]

29. (a)

$(xy)^2 = e$	Order of $xy = 2$	(M1)
$\Rightarrow (xy)(xy) = e \Rightarrow x(yx)y = e$	Associative property	(M1)(M1)
$\Rightarrow xx(yx)yy = xey$	Left and right-multiply	(M1)
$\Rightarrow e(yx)e = xy$	Order of elements given	(M1)
$\Rightarrow yx = xy$		(AG)

OR

Since x, y and xy are self-inverses, $x^{-1} = x$, $y^{-1} = y$ and $(xy)^{-1} = xy$ (R1)(R1)
Consider $xy = (xy)^{-1}$ (M1)
$= y^{-1}x^{-1}$ (M1)
$= yx$ (M1)(AG) 5

(b) Let a be any element of a group, whose identity is e.
Let a^{-1} be an inverse of a, and let b be another inverse of a
different from a^{-1}.

Now, $b = be = b(aa^{-1}) = (ba)a^{-1}$; identity and associativity properties, (M1)
then, $b = ea^{-1} = a^{-1}$, which contradicts the assumption that $b \neq a^{-1}$, (M1)
therefore there is only one inverse of a, namely a^{-1}. (R1)

OR

Let a be any element of a group whose identity is e. Let b and c be
inverses of a, so that $ab = ba = e$. (M1)
Consider $b = b(ac)$
$= (ba)c$
$= c$ (M1)
Thus any two inverses are equal, so the inverse is unique. (R1) 3

(c) If G is Abelian, then $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)\,f(y)$ and f
is an isomorphism. (M1)(R1)
If f is an isomorphism, then $f(xy) = f(x)\,f(y)$, that is,
$(xy)^{-1} = x^{-1}y^{-1} = (yx)^{-1}$
Then $xy = yx$ (M1)
and hence G is Abelian. (R1) 4

[12]