HAEF IB – FURTHER MATH HL
TEST 2
SETS, RELATIONS AND GROUPS
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Name: ______________________________________

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Questions

1. [Maximum mark: 10]
   The binary operation \( * \) is defined on \( \mathbb{N} \) by \( a * b = 1 + ab \).
   Determine whether or not \( * \)
   (a) is closed; [2 marks]
   (b) is commutative; [2 marks]
   (c) is associative; [3 marks]
   (d) has an identity element. [3 marks]

2. [Maximum mark: 8]
   The elements of sets \( P \) and \( Q \) are taken from the universal set \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). \( P = \{1, 2, 3\} \) and \( Q = \{2, 4, 6, 8, 10\} \).
   (a) Given that \( R = (P \cap Q)' \), list the elements of \( R \). [3 marks]
   (b) For a set \( S \), let \( S' \) denote the set of all subsets of \( S \),
       (i) find \( P' \); [5 marks]
       (ii) find \( n(R') \). [5 marks]
3. [Maximum mark: 13]

The function \( f : \mathbb{R} \to \mathbb{R} \) is defined by

\[
f(x) = \begin{cases} 
2x + 1 & \text{for } x \leq 2 \\
2x^2 - 2x + 5 & \text{for } x > 2.
\end{cases}
\]

(a) (i) Sketch the graph of \( f \).

(ii) By referring to your graph, show that \( f \) is a bijection.  [5 marks]

(b) Find \( f^{-1}(x) \).  [8 marks]

4. [Maximum mark: 13]

The relation \( R \) is defined on \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \) by \( aRb \) if and only if \( a(a+1) \equiv b(b+1)(\text{mod} 5) \).

(a) Show that \( R \) is an equivalence relation.  [6 marks]

(b) Show that the equivalence defining \( R \) can be written in the form

\[
(a - b)(a + b + 1) \equiv 0(\text{mod} 5).
\]

(c) Hence, or otherwise, determine the equivalence classes.  [4 marks]

5. [Maximum mark: 10]

(a) The function \( g : \mathbb{Z} \to \mathbb{Z} \) is defined by \( g(n) = |n| - 1 \) for \( n \in \mathbb{Z} \). Show that \( g \) is neither surjective nor injective.  [2 marks]

(b) The set \( S \) is finite. If the function \( f : S \to S \) is injective, show that \( f \) is surjective.  [2 marks]

(c) Using the set \( \mathbb{Z}^+ \) as both domain and codomain, give an example of an injective function that is not surjective.  [3 marks]

(d) Using the set \( \mathbb{Z}^+ \) as both domain and codomain, give an example of a surjective function that is not injective.  [3 marks]
6.  \textbf{[Maximum mark: 12]}

The binary operation $\Delta$ is defined on the set $S = \{1, 2, 3, 4, 5\}$ by the following Cayley table.

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(a) State whether $S$ is closed under the operation $\Delta$ and justify your answer. \hspace{1cm} [2]

(b) State whether $\Delta$ is commutative and justify your answer. \hspace{1cm} [2]

(c) State whether there is an identity element and justify your answer. \hspace{1cm} [2]

(d) Determine whether $\Delta$ is associative and justify your answer. \hspace{1cm} [3]

(e) Find the solutions of the equation $a\Delta b = 4\Delta b$, for $a \neq 4$. \hspace{1cm} [3]

7.  \textbf{[Maximum mark: 19]}

Consider the set $S$ defined by $S = \{s \in \mathbb{Q} : 2s \in \mathbb{Z}\}$.

You may assume that $+$ (addition) and $\times$ (multiplication) are associative binary operations on $\mathbb{Q}$.

(a) (i) Write down the six smallest non-negative elements of $S$.

(ii) Show that $\{S, +\}$ is a group.

(iii) Give a reason why $\{S, \times\}$ is not a group. Justify your answer. \hspace{1cm} [9]

(b) The relation $R$ is defined on $S$ by $s_1 R s_2$ if $3s_1 + 5s_2 \in \mathbb{Z}$.

(i) Show that $R$ is an equivalence relation.

(ii) Determine the equivalence classes. \hspace{1cm} [10]
8. [Maximum mark: 15]

Sets $X$ and $Y$ are defined by $X = ]0, 1[; Y = \{0, 1, 2, 3, 4, 5\}$.

(a) (i) Sketch the set $X \times Y$ in the Cartesian plane.

(ii) Sketch the set $Y \times X$ in the Cartesian plane.

(iii) State $(X \times Y) \cap (Y \times X)$. [5]

Consider the function $f : X \times Y \to \mathbb{R}$ defined by $f(x, y) = x + y$
and the function $g : X \times Y \to \mathbb{R}$ defined by $g(x, y) = xy$.

(b) (i) Find the range of the function $f$.

(ii) Find the range of the function $g$.

(iii) Show that $f$ is an injection.

(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(v) Find all solutions to $g(x, y) = \frac{1}{2}$. [10]