HAEF IB – FURTHER MATH HL

TEST 2

SETS, RELATIONS AND GROUPS

by Christos Nikolaidis

Marks:	**/100**
	%
Grade:	

Name:______________________________________

Date: 8/12/2017

Questions

1. *[Maximum mark: 10]*

The binary operation $*$ is defined on $\mathbb{N}$ by $a*b = 1+ab$.

Determine whether or not $*$

(a) is closed; *[2 marks]*

(b) is commutative; *[2 marks]*

(c) is associative; *[3 marks]*

(d) has an identity element. *[3 marks]*

2. *[Maximum mark: 8]*

The elements of sets P and Q are taken from the universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $P = \{1, 2, 3\}$ and $Q = \{2, 4, 6, 8, 10\}$.

(a) Given that $R = (P \cap Q')'$, list the elements of R. *[3 marks]*

(b) For a set S, let S^* denote the set of all subsets of S,

(i) find P^*;

(ii) find $n(R^*)$. *[5 marks]*

3. *[Maximum mark: 13]*

The function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases} 2x+1 & \text{for } x\le 2 \\ x^2-2x+5 & \text{for } x>2\,. \end{cases}$$

(a) (i) Sketch the graph of f.

(ii) By referring to your graph, show that f is a bijection. *[5 marks]*

(b) Find $f^{-1}(x)$. *[8 marks]*

4. *[Maximum mark: 13]*

The relation R is defined on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by aRb if and only if $a(a+1)\equiv b(b+1)(\text{mod}\,5)$.

(a) Show that R is an equivalence relation. *[6 marks]*

(b) Show that the equivalence defining R can be written in the form

$$(a-b)(a+b+1)\equiv 0(\text{mod}\,5).$$ *[3 marks]*

(c) Hence, or otherwise, determine the equivalence classes. *[4 marks]*

5. *[Maximum mark: 10]*

(a) The function $g:\mathbb{Z}\to\mathbb{Z}$ is defined by $g(n)=|n|-1$ for $n\in\mathbb{Z}$. Show that g is neither surjective nor injective. *[2 marks]*

(b) The set S is finite. If the function $f:S\to S$ is injective, show that f is surjective. *[2 marks]*

(c) Using the set $\mathbb{Z}^+$ as both domain and codomain, give an example of an injective function that is not surjective. *[3 marks]*

(d) Using the set Z^+ as both domain and codomain, give an example of a surjective function that is not injective. *[3 marks]*

6. *[Maximum mark: 12]*

The binary operation Δ is defined on the set $S = \{1, 2, 3, 4, 5\}$ by the following Cayley table.

Δ	**1**	**2**	**3**	**4**	**5**
1	1	1	2	3	4
2	1	2	1	2	3
3	2	1	3	1	2
4	3	2	1	4	1
5	4	3	2	1	5

(a) State whether S is closed under the operation Δ and justify your answer. *[2]*

(b) State whether Δ is commutative and justify your answer. *[2]*

(c) State whether there is an identity element and justify your answer. *[2]*

(d) Determine whether Δ is associative and justify your answer. *[3]*

(e) Find the solutions of the equation $a\Delta b = 4\Delta b$, for $a \neq 4$. *[3]*

7. *[Maximum mark: 19]*

Consider the set S defined by $S = \{s \in \mathbb{Q} : 2s \in \mathbb{Z}\}$.

You may assume that $+$ (addition) and $\times$ (multiplication) are associative binary operations on $\mathbb{Q}$.

(a) (i) Write down the six smallest non-negative elements of S.

(ii) Show that $\{S, +\}$ is a group.

(iii) Give a reason why $\{S, \times\}$ is not a group. Justify your answer. *[9]*

(b) The relation R is defined on S by $s_1 R s_2$ if $3s_1 + 5s_2 \in \mathbb{Z}$.

(i) Show that R is an equivalence relation.

(ii) Determine the equivalence classes. *[10]*

8. *[Maximum mark: 15]*

Sets X and Y are defined by $X =]0, 1[$; $Y = \{0, 1, 2, 3, 4, 5\}$.

(a) (i) Sketch the set $X \times Y$ in the Cartesian plane.

(ii) Sketch the set $Y \times X$ in the Cartesian plane.

(iii) State $(X \times Y) \cap (Y \times X)$. *[5]*

Consider the function $f : X \times Y \to \mathbb{R}$ defined by $f(x, y) = x + y$ and the function $g : X \times Y \to \mathbb{R}$ defined by $g(x, y) = xy$.

(b) (i) Find the range of the function f.

(ii) Find the range of the function g.

(iii) Show that f is an injection.

(iv) Find $f^{-1}(\pi)$, expressing your answer in exact form.

(v) Find all solutions to $g(x, y) = \frac{1}{2}$. *[10]*