Questions

1. [Maximum mark: 5]
   (a) Show by means of a Venn diagram that $X - Y = X \cap Y'$  
   [1 mark]
   (b) Using (a) and set algebra, prove that $A - (B \cup C) = (A - B) \cap (A - C)$  
   [4 marks]

2. [Maximum mark: 7]
   Consider the function $f : R^+ \times R \to R \times R^+$ given by
   
   $$f(x, y) = (\ln x, e^{x+y})$$

   (a) Show that $f$ is a bijection  
   [6 marks]
   (b) Find $f^{-1}$  
   [1 mark]

3. [Maximum mark: 8]
   Consider the functions $f : A \to B$ and $g : B \to C$. Given that $g \circ f : A \to C$
   is a bijection, show that
   
   (a) $f$ is an injection  
   [3 marks]
   (b) $g$ is a surjection  
   [3 marks]
   (c) $f$ and $g$ are not necessarily bijections.  
   [2 marks]
4. [Maximum mark: 15]

Let $D = R - \{1\}$ and $f : D \to D$ a function given by

$$f(x) = \frac{x+1}{x-1}$$

(a) Explain why $f$ is a bijection. [2 marks]

(b) Show that $f$ is self-inverse [2 marks]

(c) Let $T$ be a relation on $D$ given by

$$x T y \quad \text{if and only if} \quad y = f(x)$$

Determine whether $T$ is reflexive, symmetric or transitive. [5 marks]

(d) Let $S$ be a relation on $D \times R$ such that

$$(x, y) \ S \ (a, b) \quad \text{if and only if} \quad y + f(a) - b = f(x)$$

(i) Show that $S$ is an equivalence relation.

(ii) Describe the equivalence classes of $S$ (i.e. the partition of $D \times R$) [6 marks]
Consider the binary operation \( x * y = 5xy \) on the set of non-zero real numbers \( \mathbb{R}^* \).

(a) Show that \( (\mathbb{R}^*, *) \) has an identity element \( a \) and state its value. \([2 \text{ marks}]\)

(b) Show that \( (\mathbb{R}^*, *) \) is an Abelian group. \([5 \text{ marks}]\)

Consider also a homomorphism \( f : (\mathbb{R}^*, *) \rightarrow (\mathbb{R}, +) \) where \( (\mathbb{R}, +) \) is the standard additive group.

(c) Show that \( f(a) = 0 \). \([2 \text{ marks}]\)

(d) Given that \( f(x) = \ln |kx| \), where \( k \) is a positive integer

(i) find the value \( k \), by using (c)

(ii) confirm that \( f \) is a homomorphism;

(iii) explain why \( f \) is not an isomorphism;

(iv) find the kernel \( \text{Ker} f \);

(v) describe the cosets of \( \text{Ker} f \) \([6 \text{ marks}]\)
2. [Maximum mark: 20]

Consider the multiplicative group \((\mathbb{Z}_7^*, \times_7)\), where \(\mathbb{Z}_7^* = \{1,2,3,4,5,6\}\) and \(\times_7\) is the multiplication of integers modulo 7.

(a) Write down the Cayley table of this group. [4 marks]

(b) Show that \((\mathbb{Z}_7^*, \times_7)\) is cyclic and find its smallest generator. [3 marks]

Consider also the additive group \((\mathbb{Z}_6^+, +_6)\), where \(\mathbb{Z}_6 = \{0,1,2,3,4,5\}\) and \(+_6\) is the addition of integers modulo 6.

(c) If \(f\) is a homomorphism from \((\mathbb{Z}_7^*, \times_7)\) to \((\mathbb{Z}_6^+, +_6)\), with \(f(3) = 1\)

   (i) Find the value of \(f(2)\) by using the fact \(3 \times_7 3 = 2\)

   (ii) Copy and complete the following tables by applying \(f\) on the powers of 3

   \[
   \begin{array}{|c|c|c|c|c|c|c|}
   \hline
   x & 1 & 2 & 3 & 4 & 5 & 6 \\
   \hline
   f(x) & 0 & 1 & \quad & \quad & \quad & \quad \\
   \hline
   \end{array}
   \] [6 marks]

(d) If \(g\) is a homomorphism from \((\mathbb{Z}_7^*, \times_7)\) to \((\mathbb{Z}_6^+, +_6)\), with \(g(3) = 2\), copy and complete the following table

   \[
   \begin{array}{|c|c|c|c|c|c|c|}
   \hline
   x & 1 & 2 & 3 & 4 & 5 & 6 \\
   \hline
   g(x) & \quad & 2 & \quad & \quad & \quad & \quad \\
   \hline
   \end{array}
   \] [4 marks]

[2 marks]

(e) Determine which of the two functions \(f, g\) is an isomorphism. Explain. [1 marks]

(f) Write down the kernel of \(g\).