HAEF IB – FURTHER MATH HL

TEST 3

NUMBER THEORY

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Name:____________________________________

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Marks:____/90

Grade: ______

Questions

1. *[Maximum mark: 7]*

(a) Explain why $23!+5$ is not a prime number. *[1 mark]*

(b) Find all positive integers n for which $n!+5$ is a prime number. *[3 marks]*

(c) Find 2018 consecutive integers which are not prime. *[3 marks]*

2. *[Maximum mark: 8]*

Let a and b be positive integers. Show that

(a) If $a+b$ and $2a-b$ are coprime then a and b are coprime. *[2 marks]*

(b) If a and b are coprime then $\gcd(a+b, 2a-b)$ is either 1 or 3. *[4 marks]*

(c) Show by giving examples that both the results in (b) are possible. *[2 marks]*

3. *[Maximum mark: 8]*

(a) Find $2018^{2018} \pmod{13}$ *[4 marks]*

(b) Find the last digit of 2018^{2018} *[4 marks]*

4. *[maximum mark: 5]*

Show that there are infinitely many primes.

5. *[maximum mark: 7]*

Solve **analytically** the system of congruences

$$x \equiv 1 \pmod{3}$$

$$x \equiv 2 \pmod{4}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 4 \pmod{7}$$

6. *[maximum mark: 6]*

Solve the difference equation

$$u_{n+1} = 8u_n - 16u_{n-1}$$

$$u_0 = 3, \ u_1 = 16$$

7. *[maximum mark: 5]*

Show that an integer a is divisible by 3 if the sum of the digits in the expression of a in base 7 is divisible by 3.

8. *[maximum mark: 12]*

Let $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

(a) Show that

$$a + c \equiv b + d \pmod{m}$$ *[2 marks]*

$$ac \equiv bd \pmod{m}$$ *[4 marks]*

(a) Show by using mathematical induction that

$$a^n \equiv b^n \pmod{m} \quad \text{for any} \quad n \in Z^+$$ *[6 marks]*

9. *[maximum mark: 10]*

Consider the non-homogeneous difference equation

$$u_{n+2} = 5u_{n+1} - 6u_n + 10$$

$$u_1 = 8, \ u_2 = 30$$

(a) By letting $V_n = u_n - 5$, find the first 3 terms of the sequence V_n *[2 marks]*

(b) Show that

$$V_{n+2} = 5V_{n+1} - 6V_n$$ *[2 marks]*

(c) Find the general term for V_n and hence for u_n. *[6 marks]*

10. *[maximum mark: 22]*

Consider

$$d_1 = \gcd(88,136)$$

$$d_2 = \gcd(88,137)$$

$$d_3 = \gcd(88,138)$$

(a) Find the prime decomposition of the numbers 88 and 136 and hence write down the values of d_1 and $l_1 = lcm(88,136)$ *[4 marks]*

(b) Given that $9 \times 137 - 14 \times 88 = 1$, explain why $d_2 = 1$ *[2 marks]*

(c) Find the value of d_3 by using Euclid's algorithm; **Hence** express d_3 as a linear combination of 88 and 138. *[5 marks]*

(d) Solve each of the following congruences

(i) $88x \equiv 2 \pmod{136}$. *[1 mark]*

(ii) $88x \equiv 2 \pmod{137}$. *[2 marks]*

(iii) $88x \equiv 2 \pmod{138}$. *[4 marks]*

(e) Find the general solution of the Diophantine equation

$$88x - 138y = 2$$ *[4 marks]*