HAEF IB – FURTHER MATH HL

TEST 4

NUMBER THEORY

by Christos Nikolaidis

Name:_____________________________________

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Marks:____/100

Grade: ______

Questions

1. *[Maximum mark: 6]*

Let a and b be two positive integers. Show that $\gcd(a,b) \times \text{lcm}(a,b) = ab$

2. *[Maximum mark: 12]*

(a) Show that if 3 divides $(a^2 + b^2)$ then 3 divides a and 3 divides b, where $a, b \in \mathbb{Z}^+$. *[5 marks]*

(b) Show that if p is a prime number and p divides a and p divides $(a^2 + b^2)$ then p divides b, where $a, b, p \in \mathbb{Z}^+$. *[3 marks]*

(c) The greatest common divisor of x and y is denoted by (x, y). Show that if a and b are relatively prime, then $(a, bc) = (a, c)$, where $a, b, c \in \mathbb{Z}^+$. *[4 marks]*

3. *[Maximum mark: 11]*

(a) The sum of the digits of a three-digit number of the form abb is divisible by 7. Show that the number itself is divisible by 7. *[4 marks]*

(b) Use Euclid's algorithm to find the smallest positive integers x and y that satisfy the equation $57x - 13y = 7$. *[7 marks]*

4. *[Maximum mark: 6]*

Show that the product of four consecutive integers is divisible by 24.

5. *[Maximum mark: 5]*

Show that $n^4 + 4$ is not a prime for any $n > 1$, by using the binomial expansion of $(a+b)^2$

6. *[maximum mark: 6]*

(a) Find the last digit of the number 2^{2017}

(b) Find $3^{1000} \bmod 7$ by using Fermat's little theorem.

7. *[maximum mark: 6]*

Solve $88x \equiv 1 \bmod 137$

8. *[maximum mark: 10]*

Solve

$$x \equiv 1 \bmod 2$$
$$x \equiv 2 \bmod 3$$
$$x \equiv 3 \bmod 5$$

(a) By the method of the proof of the Chinese remainder theorem.
(b) By setting $x = 2k + 1$ and similar substitutions.

9. *[maximum mark: 4]*

(a) Explain why the following system does not satisfy the conditions of the Chinese remainder theorem

$$x \equiv 5 \bmod 6$$
$$x \equiv 4 \bmod 5$$
$$x \equiv 3 \bmod 4$$
$$x \equiv 2 \bmod 3$$

(b) Show that it reduces to a system that satisfies these conditions.
(Do not solve the system)

10. *[maximum mark: 7]*

(a) Show that any integral power of 10 leaves a remainder of 1 when divided by 3. *[3 marks]*

It is given that any number $y \in \mathbb{N}$ can be written in expanded form as

$$y = a_n 10^n + a_{n-1} 10^{n-1} \ldots + a_1 10 + a_0$$

(b) Show that $y = 3k$ + sum of digits of y, for some $k \in \mathbb{N}$. *[3 marks]*

(c) Show that 3 divides y if 3 divides the sum of digits of y. *[1 mark]*

11. *[maximum mark: 5]*

The population of a village is 1100 people. Show that there are at least 4 people who share the same birthday.

12. *[maximum mark: 6]*

(a) If $p_1, p_2, p_3, \ldots, p_n$ are prime numbers of the form $4m+3$, show that

$$s = 4p_1 p_2 p_3 \cdots p_n - 1$$

has a prime divisor of the form $4m+3$.

(b) Show that there are infinitely many prime numbers of the form $4m+3$.

13. *[maximum mark: 6]*

Show that for any prime number p such that $n < p < 2n$

(a) $\binom{2n}{n} \equiv 0 \bmod p$. (b) $\binom{2n}{n} \not\equiv 0 \bmod p^2$

14. *[maximum mark: 10]*

A sequence is defined recursively by

the first term $u_1 = 10$

and the recursive relation $u_{n+1} = 2u_n + 2$

(a) Given that the general solution is given by the formula $u_n = a(2)^n + b$, show that $a = 6$ and $b = -2$

(b) Prove by mathematical induction that $u_n = 6(2)^n - 2$, for $n \in \mathbb{Z}^+$