HAEF IB - MATH HL

TEST 8

PROBABILITY

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Name: **SOLUTIONS**

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Marks: ____/100

Grade: ______

Questions

1. *[Maximum mark: 7]*

   Events A and B are such that $P(A) = 0.3$ and $P(B) = 0.4$.

   (a) Find the value of $P(A \cup B)$ when

   (i) A and B are mutually exclusive;

   (ii) A and B are independent. *[4 marks]*

   (b) Given that $P(A \cup B) = 0.6$, find $P(A \mid B)$. *[3 marks]*

(a)(i) $P(A\cup B) = P(A) + P(B) = 0.3 + 0.4 = \boxed{0.7}$

(ii) $P(A\cup B) = P(A) + P(B) - P(A)\cdot P(B)$

$= 0.3 + 0.4 - 0.12 = \boxed{0.58}$

(b) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow 0.6 = 0.3 + 0.4 - P(A\cap B)$

$\Rightarrow P(A\cap B) = 0.1$

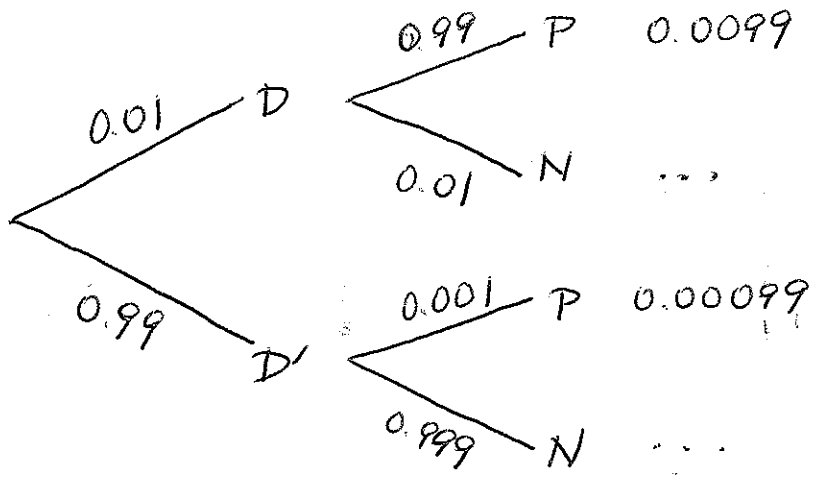
Then $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{0.1}{0.4} = \dfrac{1}{4} = \boxed{0.25}$

2. *[Maximum mark: 6]*

In a population of rabbits, 1 % are known to have a particular disease. A test is developed for the disease that gives a positive result for a rabbit that does have the disease in 99 % of cases. It is also known that the test gives a positive result for a rabbit that does not have the disease in 0.1 % of cases. A rabbit is chosen at random from the population.

(a) Find the probability that the rabbit tests positive for the disease. *[3 marks]*

(b) Given that the rabbit tests positive for the disease, show that the probability that the rabbit does not have the disease is less than 10 %. *[3 marks]*



(a) $P(\text{positive}) = 0.0099 + 0.00099 = \boxed{0.01089}$

(b) $P(D' \mid \text{positive}) = \dfrac{0.00099}{0.01089} < \dfrac{0.001}{0.01} = \boxed{0.1}$ (10%)

3. *[Maximum mark: 7]*

Two players, A and B, alternately throw a fair six–sided dice, with A starting, until one of them obtains a six. Find the probability that B obtains the first six.

For B

$P(\text{six in } 1^{st} \text{ throw}) = \frac{5}{6} \cdot \frac{1}{6}$

$P(\text{six in } 2^{nd} \text{ throw}) = \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6}$

$P(\text{six in } 3^{rd} \text{ throw}) = \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6}$ ...

G.S

$u_1 = \frac{5}{36}$

$r = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$

$P(B \text{ wins}) = S_\infty = \dfrac{\frac{5}{36}}{1 - \frac{25}{36}} = \dfrac{\frac{5}{36}}{\frac{11}{36}}$

$= \boxed{\frac{5}{11}}$

4. *[Maximum mark: 10]*

The probability distribution of a discrete random variable X is defined by

$$P(X=x) = k\binom{3}{x}, \qquad x = 0, 1, 2, 3.$$

(a) Find the value of k *[3 marks]*

(b) Find $E(X)$ *[2 marks]*

(c) Find $Var(X)$ *[3 marks]*

(d) Find the median. *[2 marks]*

In fact,

X	0	1	2	3
$P(X=x)$	k	$3k$	$3k$	k

(a) $k+3k+3k+k=1 \Rightarrow 8k=1 \Rightarrow \boxed{k=\frac{1}{8}}$

(b) $E(X)=0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=\frac{12}{8}=\boxed{\frac{3}{2}}$

(c) $E(X^2)=0^2\times\frac{1}{8}+1^2\times\frac{3}{8}+2^2\cdot\frac{3}{8}+3^2\times\frac{1}{8}=\frac{24}{8}=3$

$Var(X)=3-\left(\frac{3}{2}\right)^2=3-\frac{9}{4}=\boxed{\frac{3}{4}}$

(d) Due to symmetry Median $=\boxed{1.5}$

5. *[Maximum mark: 16]*

A random variable has a probability density function given by

$$f(x) = \begin{cases} kx(2-x), & 0 \le x \le 1 \\ k & 1 \le x \le 2 \end{cases}$$

(a) Show that $k = \frac{3}{5}$ *[2 marks]*

(b) Find $P(X \ge 0.5)$ *[2 marks]*

(c) Find $E(X)$ *[3 marks]*

(d) Find $Var(X)$ *[3 marks]*

(e) Find the median. *[3 marks]*

(f) Find the interquartile range. *[3 marks]*

(a) $k\int_0^1 x(2-x)\,dx + k\int_1^2 dx = 1 \Rightarrow \frac{2}{3}k + k = 1 \Rightarrow \frac{5}{3}k = 1$

$\Rightarrow \boxed{k = \frac{3}{5}}$

(b) $P(X \le 0.5) = \int_0^{0.5} \frac{3}{5}x(2-x)\,dx = \frac{1}{8}$

Hence $\boxed{P(X \ge 0.5) = \frac{7}{8}}$

(c) $E(X) = \int_0^1 \frac{3}{5}x^2(2-x)\,dx + \int_1^2 \frac{3}{5}x\,dx = \frac{1}{4} + \frac{9}{10} = \frac{23}{20} = \boxed{1.15}$

(d) $E(X^2) = \int_0^1 \frac{3}{5}x^3(2-x)\,dx + \int_1^2 \frac{3}{5}x^2\,dx = \frac{9}{50} + \frac{7}{5} = \frac{79}{50} = \underline{\underline{1.58}}$

$Var(X) = 1.58 - 1.15^2 = \boxed{0.2575}$

(e) $P(X \le 1) = \frac{2}{5}$. $\int_M^2 \frac{3}{5}\,dx = \frac{1}{2} \Rightarrow \frac{3}{5}(2-M) = \frac{1}{2} \Rightarrow \boxed{M = \frac{7}{6}}$

(f) $\frac{3}{5}\int_0^{Q_1} 2x - x^2\,dx = \frac{1}{4} \Rightarrow \left[x^2 - \frac{x^3}{3}\right]_0^{Q_1} = \frac{5}{12} \Rightarrow Q_1^2 - \frac{Q_1^3}{3} = \frac{5}{12} \Rightarrow \underline{\underline{Q_1 = 0.744}}$

$\frac{3}{5}\int_{Q_3}^2 dx = \frac{1}{4} \Rightarrow \frac{3}{5}(2 - Q_3) = \frac{1}{4} \Rightarrow \underline{\underline{Q_3 = \frac{19}{12} = 1.583}}$ $\boxed{IQR = 0.839}$

6. *[Maximum mark: 6]*

Find $E(X^2)$ in each of the following cases

(a) X follows $B(10,0.3)$ *[2 marks]*

(b) X follows $N(10,0.3)$ *[2 marks]*

(c) X follows $Po(10)$ *[2 marks]*

$Var(X) = E(X^2) - E(X)^2 \Rightarrow E(X^2) = E(X)^2 + Var(X)$

(a) $E(X) = np = 10 \times 0.3 = 3$ $\quad Var(X) = np(1-p) = 3 \times 0.7 = 2.1$

$E(X^2) = 3^2 + 2.1 = \boxed{11.1}$

(b) $E(X^2) = \mu^2 + \sigma^2 = 100 + 0.3 = \boxed{100.3}$

(c) $E(X^2) = 10^2 + 10 = \boxed{110}$

7. *[Maximum mark: 6]*

A biased coin is weighted such that the probability of obtaining a head is $\frac{4}{7}$.

The coin is tossed 6 times and X denotes the number of heads observed.

(a) Find **analytically** the value of the ratio . $\frac{P(X=3)}{P(X=2)}$ *[3 marks]*

(b) Find the probability that more heads than tails are observed. *[3 marks]*

(a) $\frac{P(X=3)}{P(X=2)} = \frac{\binom{6}{3}\left(\frac{4}{7}\right)^3\left(\frac{3}{7}\right)^3}{\binom{6}{2}\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)^4} = \frac{\frac{6!}{3!3!}}{\frac{6!}{2!4!}} \cdot \frac{\frac{4}{\cancel{7}}}{\frac{3}{\cancel{7}}} = \frac{4}{3} \cdot \frac{4}{3} = \boxed{\frac{16}{9}}$

(b) $P(X \geq 4) = 1 - P(X \leq 3)$

$= 1 - 0.51476 = \boxed{0.485}$

8. *[Maximum mark: 8]*

The fish in a lake have weights that are normally distributed with a mean of 1.3 kg and a standard deviation of 0.2 kg.

(a) Determine the probability that a fish that is caught weighs less than 1.4 kg. *[1 mark]*

(b) 25% of the fish weigh more than a. Find the value of a. *[2 marks]*

(c) John catches 6 fish. Calculate the probability that at least 4 of the fish weigh more than 1.4 kg. *[3 marks]*

(d) Determine the probability that a fish that is caught weighs less than 1 kg, given that it weighs less than 1.4 kg. *[2 marks]*

$\mu = 1.3 \quad \sigma = 0.2$

(a) $P(X < 1.4) = \boxed{0.691}$ $\boxed{\text{Ncd}}$

(b) $P(X > a) = 0.25 \Rightarrow \boxed{a = 1.43}$ $\boxed{\text{InvN}}$

(c) $P(X > 1.4) = 0.309$

BINOMIAL WITH $n = 6 \quad p = 0.309$

$P(Y \geq 4) = 1 - P(X \leq 3) = 1 - 0.922 = \boxed{0.078}$

(d) $P(X < 1 \mid X < 1.4) = \dfrac{P(X < 1)}{P(X < 1.4)} = \dfrac{0.0668}{0.691} = \boxed{0.0967}$

9. *[Maximum mark: 5]*

The speeds of cars at a certain point on a straight road are normally distributed with mean μ and standard deviation σ. 15% of the cars travelled at speeds greater than 90 km h^{-1} and 15 % of them at speeds less than 50 km h^{-1}. Find μ and σ.

Due to symmetry $\mu = \frac{50+90}{2} = \boxed{70}$

$P(X < 50) = 0.15$

$$Z = \frac{X-\mu}{\sigma} \Rightarrow \frac{50-70}{\sigma} = -1.0364$$

$$\Rightarrow \sigma = \frac{20}{1.0364} \Rightarrow \boxed{\sigma = 19.3}$$

10. *[Maximum mark: 7]*

Mr Lee is planning to go fishing this weekend. Assuming that the number of fish caught per hour follows a Poisson distribution with mean 0.6, find

(a) the probability that he catches at least one fish in the first hour; *[2 marks]*

(b) the probability that he catches exactly three fish if he fishes for four hours; *[2 marks]*

(c) the number of **complete** hours that Mr Lee needs to fish so that the probability of catching more than two fish exceeds 80 %. *[3 marks]*

(a) $m=0.6 \quad P(X \geq 1) = 1 - P(X=0)$

$= 1 - 0.5488$

$= \boxed{0.451}$

(b) $m = 4 \times 0.6 = 2.4$

$P(X=3) = \boxed{0.209}$

(c) $P(X>2) > 0.8 \Rightarrow P(X \leq 2) < 0.2$

$\Rightarrow e^{-m} + me^{-m} + \frac{m^2}{2}e^{-m} < 0.2$

By GDC $m > 4.28$

hours $t > \frac{4.28}{0.6} = 7.13 \Rightarrow \boxed{t = 8 \text{ hours}}$

11. *[Maximum mark: 6]*

The random variable X follows a Poisson distribution with mean λ.

(a) Find λ if $P(X < 2) = 0.123$. *[4 marks]*

(b) With this value of λ, find $P(0 < X < 9)$. *[2 marks]*

(a) $e^{-\lambda} + \lambda e^{-\lambda} = 0.123$ $\boxed{\lambda = 3.63}$

(b) $P(0 < X < 9) = P(1 \leq X \leq 8)$

$= 0.9877 - 0.0265$

$= \boxed{0.9612}$

12. *[Maximum mark: 16]*

Tim throws two identical fair dice simultaneously. Each die has six faces: two faces numbered 1, two faces numbered 2 and two faces numbered 3. His score is the sum of the two numbers shown on the dice.

(a) (i) Calculate the probability that Tim obtains a score of 6.

(ii) Calculate the probability that Tim obtains a score of at least 3. *[3 marks]*

Tim plays a game with his friend Bill, who also has two dice numbered in the same way. Bill's score is the sum of the two numbers shown on his dice.

(b) (i) Calculate the probability that Tim and Bill **both** obtain a score of 6.

(ii) Calculate the probability that Tim and Bill obtain the same score. *[4 marks]*

(c) Let X denote the largest number shown on the four dice.

(i) Show that $P(X \leq 2) = \frac{16}{81}$.

(ii) Copy and complete the following probability distribution table.

x	1	2	3
$P(X = x)$	$\frac{1}{81}$		

(iii) Calculate $E(X)$ *[5 marks]*

(d) Given that $X = 3$, find the probability that the sum of the numbers shown on the four dice is 8. *[4 marks]*

(a) (i) $P(\text{sum} = 6) = P(3\text{-}3) = \frac{1}{3} \times \frac{1}{3} = \boxed{\frac{1}{9}}$

(ii) $P(\text{sum} \geq 3) = 1 - P(\text{sum} = 2) = 1 - \frac{1}{3} \times \frac{1}{3} = \boxed{\frac{8}{9}}$

(b) (i) $P(\text{both scores } 6) = \frac{1}{9} \times \frac{1}{9} = \boxed{\frac{1}{81}}$

(ii) $P(\text{same score}) = \left(\frac{1}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{3}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{1}{9}\right)^2 = \boxed{\frac{19}{81}}$

(c) (i) $P(\text{one die} \le 2) = P(1 \text{ or } 2) = \frac{2}{3}$

$P(\text{for four dice}) = P(X \le 2) = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

(ii)

x	1	2	3
$P(X=x)$	$\frac{1}{81}$	$\frac{15}{81}$	$\frac{65}{81}$
Cumulative	$\frac{1}{81}$	$\frac{16}{81}$	$\frac{81}{81}$

(iii) $E(X) = 1 \times \frac{1}{81} + 2 \times \frac{15}{81} + 3 \times \frac{65}{81} = \boxed{\frac{226}{81}} \cong \boxed{2.79}$

(d) $P(\text{sum}=8 \mid X=3) = \dfrac{P(\text{sum}=8 \text{ and } X=3)}{P(X=3)}$

Cases where $X=3$ and sum$=8$:

1 1 3 3 $\binom{4}{2}\binom{2}{2} = 6$

1 2 2 3 $\binom{4}{2}\binom{2}{1}\binom{1}{1} = 12$

Thus $P(\text{sum}=8 \text{ and } X=3) = \frac{18}{81}$

Therefore,

$P(\text{sum}=8 \mid X=3) = \dfrac{\frac{18}{81}}{\frac{65}{81}} = \boxed{\frac{18}{65}}$