Questions

1. [maximum mark: 8]

Let \( \mathbf{u} = \begin{pmatrix} a \\ 2 \\ 3b + 2 \end{pmatrix} \) and \( \mathbf{v} = \begin{pmatrix} -3 \\ 1 \\ b \end{pmatrix} \)

Find the values of scalars \( a \) and \( b \) in each of the following cases

(a) \( \mathbf{u} \) is perpendicular to \( \mathbf{v} \), and \( a = 3b \) [5 marks]

(b) \( \mathbf{u} \) is parallel to \( \mathbf{v} \) [3 marks]

2. [maximum mark: 10]

Consider the parallelogram OABC. Let

\( \mathbf{a} \) be the position vector of point A

\( \mathbf{c} \) be the position vector of point C.

(a) Express \( \mathbf{OB} \) and \( \mathbf{CA} \) in terms of \( \mathbf{a} \) and \( \mathbf{c} \) [2 marks]

(b) Show that if \( |\mathbf{OB}| = |\mathbf{CA}| \) then \( \mathbf{a} \perp \mathbf{c} \) [3 marks]

(c) Show that if \( \mathbf{OB} \perp \mathbf{CA} \) then \( |\mathbf{a}| = |\mathbf{c}| \) [3 marks]

(d) If both the assumptions of (b) and (c) hold, describe geometrically the result. [2 marks]
3. [maximum mark: 6]
The Cartesian equations of the line \( l \) are
\[
\frac{x - 1}{-1} = \frac{y - 3}{2} = \frac{z - 6}{-1}
\]
(a) Write down the vector equation of the line \( l \)  
[1 mark]
(b) Show that the point \( A(4, -2, 5) \) does not lie in the line \( l \).  
[1 mark]
(c) Find the Cartesian equation of the plane containing line \( l \) and the point \( A \).  
[4 marks]

4. [maximum mark: 14]
Consider the points \( A(1, 2, 1), B(0, -1, 2), C(1, 0, 2) \) and \( D(2, -1, -6) \).
(a) Calculate \( \overrightarrow{AB} \times \overrightarrow{BC} \).  
[3 marks]
(b) Find the area of the triangle \( ABC \).  
[2 marks]
(c) Find the Cartesian equation of the plane \( \Pi \) containing the points \( A, B \) and \( C \).  
[2 marks]
(d) Find the distance of the point \( C \) to the line passing through \( A \) and \( B \)  
[3 marks]
(e) Find the distance from the point \( D \) to the plane \( \Pi \).  
[4 marks]

5. [maximum mark: 12]
Consider the system of simultaneous equations.
\[
\begin{align*}
  x - 2y - az &= b \\
  2x - y + 3z &= 2 \\
  3x + y + 2z &= -2
\end{align*}
\]
(a) Find the values of \( a \) and \( b \) for which the system has a unique solution.  
[5 marks]
(b) Find the values of \( a \) and \( b \) for which the system has no solution.  
[2 marks]
(c) Find the values of \( a \) and \( b \) for which the system has infinitely many solutions.  
Find the general solution.  
[3 marks]
(d) Give a geometric description in cases (b) and (c).  
[2 marks]
Questions

1. [maximum mark: 16]
Consider the lines

\[ L_1: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \]
\[ L_2: \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \]

and the planes

\[ \Pi_1: 2x + 3y + z = 7 \]
\[ \Pi_2: 4x + 7y + 4z = 19 \]

Find

(a) The angle between the lines \( L_1 \) and \( L_2 \) [2 marks]
(b) The angle between the planes \( \Pi_1 \) and \( \Pi_2 \) [2 marks]
(c) The angle between the line \( L_1 \) and the plane \( \Pi_2 \) [2 marks]
(d) The point of intersection of the lines \( L_1 \) and \( L_2 \) [4 marks]
(e) The point of intersection of the line \( L_1 \) and plane \( \Pi_1 \) [3 marks]
(f) The line of intersection of the planes \( \Pi_1 \) and \( \Pi_2 \) [3 marks]

2. [maximum mark: 8]
(a) Solve the following systems of simultaneous equations:

\[
\begin{align*}
2x + 3y - z &= 2 \\
2x + 3y - z &= 2 \\
3x + 2y + 5z &= 5 \\
\end{align*}
\]
\[
\begin{align*}
x - y + 2z &= 1 \\
x - y + 2z &= 5 \\
3x + 2y + 6z &= 6 \\
\end{align*}
\]
\[
\begin{align*}
x - y + 2z &= 5 \\
3x + 2y + 7z &= 7 \\
\end{align*}
\]

[5 marks]

(b) Give a geometric interpretation of the solutions. [3 marks]
3. [maximum mark: 7]

Find the distance between the two points on the line \( L_1: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \) whose distance from point \( A(1,1,-1) \) is \( 3\sqrt{5} \) units. [7 marks]

4. [maximum mark: 10]

Consider the lines

\[
L_1: \frac{x-2}{-2} = 1 - y = 2z \quad L_2: x = 2y, z = 3
\]

(a) Show that the lines \( L_1 \) and \( L_2 \) intersect at the point \( A(-10,-5,3) \) [2 marks]

Let \( L_1' \) be the reflection of the line \( L_1 \) in the line \( L_2 \)

(b) Find the angle between the line \( L_1 \) and \( L_1' \) [4 marks]

(c) Find the equation of the line \( L_1' \). [4 marks]

5. [maximum mark: 9]

Consider the points \( A(1, 2, 1) \) and \( B(3, 0, 5) \).

(a) Find the Cartesian equation of the plane \( \Pi \) which consists of the equidistant points from \( A \) and \( B \). [4 marks]

(b) Show that the points \( A \) and \( B \) lie in the plane \( \Pi: x + 3y + z = 8 \) and find the equation of the line \( L \) on the plane \( \Pi \) which consists of the equidistant points from \( A \) and \( B \). [5 marks]