HAEF IB - MATH HL

TEST 8a

COMPLEX NUMBERS

by Christos Nikolaidis

Name:____________________________________

Date:________________

Marks:____/80

Grade: ______

Questions

1. *[Maximum mark: 5]*

Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$. *[5 marks]*

2. *[Maximum mark: 7]*

Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$; *[2 marks]*

(b) $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0, z \neq -1$. *[5 marks]*

3. *[Maximum mark: 6]*

Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.
Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d. *[6 marks]*

4. *[Maximum mark: 14]*

The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$. . *[2 marks]*

(b) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form. *[3 marks]*

(c) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. *[5 marks]*

(d) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \,\mathrm{d}\theta$. *[4 marks]*

5. *[Maximum mark: 10]*

(a) Solve the equation

$$z^3 = 32 + 32\sqrt{3}i$$

giving your answer in the form $re^{i\theta}$ where $r > 0, -\pi < \theta \leq \pi$. *[6 marks]*

(b) Show that your solutions satisfy the equation

$$z^9 + 2^k = 0$$

for an integer *k*, the value of which should be stated. . *[4 marks]*

6. *[Maximum mark: 14]*

Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$. *[4 marks]*

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram. *[4 marks]*

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where *z* is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω. *[6 marks]*

7. *[Maximum mark: 12]*

(a) Use Euler's form for $e^{i\theta}$ and $e^{-i\theta}$ to show that

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

and find a similar expression for $\sin\theta$. *[2 marks]*

(b) **Hence**, by using binomial theorem, show that

$$\cos^2\theta\sin^4\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$ *[6 marks]*

(c) **Hence** find the exact value of

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta d\theta.$$ *[4 marks]*

8. *[Maximum mark: 12]*

The integrals *C* and *S* are defined by

$$C = \int_0^{\frac{\pi}{2}} e^{2x}\cos 3x dx \qquad \text{and} \qquad S = \int_0^{\frac{\pi}{2}} e^{2x}\sin 3x dx.$$

By considering $C + \mathrm{i}S$ as a single integral, show that

$$C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$$

and obtain a similar expression for *S*. *[12 marks]*

[You may assume that the standard result for $\int e^{kx}dx$ *remains true when k is a complex constant,*

so that $\int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x} + c$*]*