HAEF IB - MATH HL
TEST 8b
COMPLEX NUMBERS

by Christos Nikolaidis

Name:___________________________________

Date:________________

Marks:____/80
Grade: ______

Questions

1. *[Maximum mark: 6]*

The two complex numbers $z = \dfrac{a}{1+i}$ and $w = \dfrac{b}{1-2i}$ where $a, b \in R$ are such that $z + w = 3$. Calculate the value of a and of b *[6 marks]*

2. *[Maximum mark: 7]*

The complex number z is given by :

$$z = -\sqrt{3} + i$$

(a) Find the value of the modulus and the argument of z. *[3 marks]*

(b) Find $\arg\left(\dfrac{i}{z}\right)$, giving your answer in degrees *[4 marks]*

3. *[Maximum mark: 6]*

Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.

Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d. *[6 marks]*

4. *[Maximum mark: 10]*

(a) Solve the equation

$$z^3 = 32 + 32\sqrt{3}\, i$$

giving your answer in the form $re^{i\theta}$ where $r > 0, -\pi < \theta \leq \pi$. *[6 marks]*

(b) Show that your solutions satisfy the equation

$$z^9 + 2^k = 0$$

for an integer k, the value of which should be stated. *[4 marks]*

5. *[Maximum mark: 7]*

(a) Show that $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\frac{n\pi}{4}$, $n \in N$ *[5 marks]*

(b) **Hence** find that the value of $(1+i)^{19} + (1-i)^{19}$, simplified to an integer. *[2 marks]*

6. *[Maximum mark: 18]*

Consider $z = \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$

(a) Show that z is a 7th root of 1. *[2 marks]*

(b) Write down all the 7th roots of 1. *[3 marks]*

(c) Show that the sum of the 7th roots of 1 is 0. *[2 marks]*

(d) Hence, or otherwise, show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ *[4 marks]*

Let

$$A = z + z^2 + z^4 \quad \text{and} \quad B = z^3 + z^5 + z^6$$

(e) Given $A + B = m$ and $AB = n$, where m and n are integers, find

(i) the value of m; (ii) the value of n. *[4 marks]*

(f) Find $A^2 + B^2$ *[3 marks]*

7. *[Maximum mark: 13]*

(a) Show that $\left|e^{i\theta}\right| = 1$ *[1 mark]*

Consider the geometric series $1 + \frac{1}{3}e^{i\theta} + \frac{1}{9}e^{2i\theta} + \cdots$

(b) Write down the common ratio, z, of the series, and state its modulus. *[2 marks]*

(c) Find an expression for the sum to infinity of the series. *[2 marks]*

(d) Hence show that $\sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta + \cdots = \frac{9\sin\theta}{10 - 6\cos\theta}$ *[8 marks]*

8. *[Maximum mark: 13]*

(a) Use Euler's form for $e^{i\theta}$ and $e^{-i\theta}$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

and find a similar expression for $\sin\theta$. *[2 marks]*

(b) **Hence**, by using the binomial theorem, show that

$$\cos^2\theta\sin^4\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$ *[7 marks]*

(c) **Hence** find the exact value of

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta d\theta.$$ *[4 marks]*