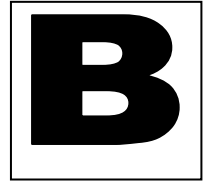


HAEF IB - MATH HL
TEST 7
COMPLEX NUMBERS

by Christos Nikolaidis



Name: _____

Date: _____

Marks: ____/100
Grade: _____

Questions

1. Solve the following equation for z , where z is a complex number.

$$\frac{z}{3+4i} + \frac{z-1}{5i} = \frac{5}{3-4i}$$

Give your answers in the form $a + bi$ where $a, b \in \mathbb{Z}$.

(Total 6 marks)

2. Find the values of n such that $(1 + \sqrt{3}i)^n$ is a real number.

(Total 5 marks)

3. Consider the equation $z^3 + az^2 + bz + c = 0$, where $a, b, c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is $-1 + 3i$, find

- (a) the other two roots; (4)
(b) a, b and c . (4)

(Total 8 marks)

4. (a) Solve the equation $z^3 = -2 + 2i$, giving your answers in modulus–argument form. (6)
(b) **Hence** show that one of the solutions is $1 + i$ when written in Cartesian form. (2)

(Total 8 marks)

5. (a) Write down the expansion of $(\cos \theta + i \sin \theta)^5$ in the form $a + ib$, where a and b are in terms of $\sin \theta$ and $\cos \theta$. (4)
(b) Hence show that $\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$. (3)
(c) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$$\cos \frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}} \text{ and state the value of } \cos \frac{7\pi}{10}. \quad (8)$$

(Total 15 marks)

6. The complex number z is defined as $z = \cos \theta + i \sin \theta$.

(a) Show that $z^n - \frac{1}{z^n} = 2i \sin(n\theta)$. (4)

(b) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form. (3)

(c) Hence show that $16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta$. (5)

(d) Find $\int_0^{\pi/2} \sin^5 \theta \, d\theta$. (4)

(Total 16 marks)

7. Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \dots$

(a) Find an expression for z , the common ratio of this series. (2)

(b) Write down the value of $|z|$. (1)

(c) Write down an expression for the sum to infinity of this series. (2)

(d) (i) Express your answer to part (c) in terms of $\sin \theta$ and $\cos \theta$.

(ii) Hence show that

$$\cos \theta + \frac{1}{2} \cos 2\theta + \frac{1}{4} \cos 3\theta + \dots = \frac{4 \cos \theta - 2}{5 - 4 \cos \theta}. \quad (10)$$

(Total 15 marks)

8. The roots of the equation $z^3 - 1 = 0$ are defined by $1, \omega$ and ω^2 .

(a) Sketch an Argand diagram to show these roots. (2)

(b) Show that $1 + \omega + \omega^2 = 0$. (2)

(c) Hence evaluate

(i) $(2 + \omega)(2 + \omega^2)$

(ii) $\frac{1}{2 + \omega} + \frac{1}{2 + \omega^2}$ (5)

(d) Find a cubic equation, with integer coefficients, which has roots

$$2, \quad \frac{1}{2 + \omega} \quad \text{and} \quad \frac{1}{2 + \omega^2} \quad (6)$$

(Total 15 marks)

9. The integrals C and S are defined by

$$C = \int_0^{\pi/2} e^{2x} \cos 3x \, dx \quad \text{and} \quad S = \int_0^{\pi/2} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\frac{1}{13}(2 + 3e^{\pi})$$

and obtain a similar expression for S .

[You may assume that the standard result for $\int e^{kx} \, dx$ remains true when k is a

complex constant, so that $\int e^{(a+ib)x} \, dx = \frac{1}{a+ib} e^{(a+ib)x} + \text{constant}$]

(Total 12 marks)