HAEF IB - MATH HL

TEST 7

COMPLEX NUMBERS

by Christos Nikolaidis

B

Name:____________________________________

Date:________________

Marks:____**/100**

Grade: ______

Questions

1. Solve the following equation for z, where z is a complex number.

$$\frac{z}{3+4i}+\frac{z-1}{5i}=\frac{5}{3-4i}$$

Give your answers in the form $a+bi$ where $a,b \in Z$.

(Total 6 marks)

2. Find the values of n such that $(1+\sqrt{3}\,i)^n$ is a real number.

(Total 5 marks)

3. Consider the equation $z^3 + az^2 + bz + c = 0$, where $a, b, c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is $-1 + 3i$, find

(a) the other two roots; **(4)**

(b) a, b and c. **(4)**

(Total 8 marks)

4. (a) Solve the equation $z^3 = -2 + 2i$, giving your answers in modulus–argument form. **(6)**

(b) **Hence** show that one of the solutions is $1 + i$ when written in Cartesian form. **(2)**

(Total 8 marks)

5. (a) Write down the expansion of $(\cos\theta + i\sin\theta)^5$ in the form $a + ib$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. **(4)**

(b) Hence show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **(3)**

(c) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$\cos\frac{\pi}{10}=\sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$. **(8)**

(Total 15 marks)

6. The complex number z is defined as $z = \cos\theta + i\sin\theta$.

(a) Show that $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$. **(4)**

(b) Use the binomial theorem to expand $\left(z - \frac{1}{z}\right)^5$ giving your answer in simplified form. **(3)**

(c) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. **(5)**

(d) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta$. **(4)**

(Total 16 marks)

7. Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

(a) Find an expression for z, the common ratio of this series. **(2)**

(b) Write down the value of $|z|$. **(1)**

(c) Write down an expression for the sum to infinity of this series. **(2)**

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$ **(10)**

(Total 15 marks)

8. The roots of the equation $z^3 - 1 = 0$ are defined by 1, ω and ω^2.

(a) Sketch an Argand diagram to show these roots. **(2)**

(b) Show that $1 + \omega + \omega^2 = 0$. **(2)**

(c) **Hence** evaluate

(i) $(2+\omega)(2+\omega^2)$

(ii) $\frac{1}{2+\omega} + \frac{1}{2+\omega^2}$ **(5)**

(d) Find a cubic equation, with integer coefficients, which has roots

$$2, \quad \frac{1}{2+\omega} \quad \text{and} \quad \frac{1}{2+\omega^2}$$ **(6)**

(Total 15 marks)

9. The integrals C and S are defined by

$$C = \int_0^{\frac{\pi}{2}} e^{2x}\cos 3x\,dx \quad \textbf{and} \quad S = \int_0^{\frac{\pi}{2}} e^{2x}\sin 3x\,dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$$

and obtain a similar expression for S.

[You may assume that the standard result for $\int e^{kx}\,dx$ remains true when k is a complex constant, so that $\int e^{(a+ib)x}\,dx = \frac{1}{a+ib}e^{(a+ib)x} + \text{constant}$.]

(Total 12 marks)