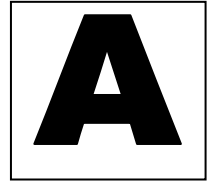


**HAEF IB - MATH HL**  
**TEST 7**  
**COMPLEX NUMBERS**

*by Christos Nikolaidis*



Name: \_\_\_\_\_

Date: \_\_\_\_\_

**Marks: \_\_\_\_/100**  
**Grade: \_\_\_\_\_**

**Questions**

1. Given that  $\frac{z}{z+2} = 2 - i$ ,  $z \in \mathbb{C}$ , find  $z$  in the form  $a + ib$ .

**(Total 4 marks)**

2. Given that  $z = \cos\theta + i \sin\theta$  show that

(a)  $\operatorname{Im}\left(z^n + \frac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$ ; **(2)**

(b)  $\operatorname{Re}\left(\frac{z-1}{z+1}\right) = 0, z \neq -1$ . **(5)**

**(Total 7 marks)**

3. Consider the polynomial  $p(x) = x^4 + ax^3 + bx^2 + cx + d$ , where  $a, b, c, d \in \mathbb{R}$ .  
Given that  $1 + i$  and  $1 - 2i$  are zeros of  $p(x)$ , find the values of  $a, b, c$  and  $d$ .

**(Total 7 marks)**

- 4.

(a) Solve the equation

$$z^3 = (32 + 32\sqrt{3}i),$$

giving your answers in the form  $re^{i\theta}$  where  $r > 0, -\pi < \theta \leq \pi$ . **(6 marks)**

(b) Show that your solutions satisfy the equation

$$z^9 + 2^8 = 0,$$

for an integer  $k$ , the value of which should be stated. **(5 marks)**

**(Total 11 marks)**

5. (a) Write down the expansion of  $(\cos\theta + i \sin\theta)^3$  in the form  $a + ib$ , where  $a$  and  $b$  are in terms of  $\sin\theta$  and  $\cos\theta$ . **(2)**

(b) Hence show that  $\cos 3\theta = 4 \cos^3\theta - 3 \cos\theta$ . **(3)**

(c) Similarly show that  $\cos 5\theta = 16 \cos^5\theta - 20 \cos^3\theta + 5 \cos\theta$ . **(4)**

(d) Hence solve the equation

$$\cos 5\theta + \cos 3\theta + \cos\theta = 0, \quad \text{where } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad \text{(6)}$$

**(Total 15 marks)**

6. The complex number  $z$  is defined as  $z = \cos \theta + i \sin \theta$ .
- (a) Show that  $z^n - \frac{1}{z^n} = 2i \sin(n\theta)$ . (4)
- (b) Use the binomial theorem to expand  $\left(z - \frac{1}{z}\right)^5$  giving your answer in simplified form. (3)
- (c) Hence show that  $16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta$ . (5)
- (d) Find  $\int_0^{\frac{\pi}{2}} \sin^5 \theta \, d\theta$ . (4)

(Total 16 marks)

7. (a) By expressing  $\cos \theta$  and  $\sin \theta$  in terms of  $e^{i\theta}$  and  $e^{-i\theta}$ , show that
- $$\cos^2 \theta \sin^4 \theta = \frac{1}{32} (\cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2). \quad (8)$$
- (b) Hence find the exact value of
- $$\int_0^{\frac{\pi}{3}} \cos^2 \theta \sin^4 \theta \, d\theta. \quad (4)$$

(Total 12 marks)

8. Consider  $\omega = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$ .
- (a) Show that
- (i)  $\omega^3 = 1$ ;
- (ii)  $1 + \omega + \omega^2 = 0$ . (4)
- (b) (i) Deduce that  $e^{i\theta} + e^{i\left(\theta + \frac{2\pi}{3}\right)} + e^{i\left(\theta + \frac{4\pi}{3}\right)} = 0$ .
- (ii) Illustrate this result for  $\theta = \frac{\pi}{2}$  on an Argand diagram. (4)
- (c) (i) Expand and simplify  $F(z) = (z - 1)(z - \omega)(z - \omega^2)$  where  $z$  is a complex number.
- (ii) Solve  $F(z) = 7$ , giving your answers in terms of  $\omega$ . (8)

(Total 16 marks)

9. The integrals  $C$  and  $S$  are defined by

$$C = \int_0^{\frac{\pi}{2}} e^{2x} \cos 3x \, dx \quad \text{and} \quad S = \int_0^{\frac{\pi}{2}} e^{2x} \sin 3x \, dx.$$

By considering  $C + iS$  as a single integral, show that

$$C = -\frac{1}{13} (2 + 3e^{\pi})$$

and obtain a similar expression for  $S$ .

*[You may assume that the standard result for  $\int e^{kx} \, dx$  remains true when  $k$  is a*

*complex constant, so that  $\int e^{(a+ib)x} \, dx = \frac{1}{a+ib} e^{(a+ib)x} + \text{constant}$ ]*

(Total 12 marks)