A

HAEF IB - MATH HL

TEST 7

COMPLEX NUMBERS

by Christos Nikolaidis

Name:____________________________________

Date:________________

Marks:____/100

Grade: ______

Questions

1. Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

(Total 4 marks)

2. Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$; **(2)**

(b) $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0, z \neq -1$. **(5)**

(Total 7 marks)

3. Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.
Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d.

(Total 7 marks)

4.

(*a*) Solve the equation

$$z^3 = (32 + 32\sqrt{3}\mathrm{i}),$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$, $-\pi < \theta \leqslant \pi$. **(6 marks)**

(*b*) Show that your solutions satisfy the equation

$$z^9 + 2^k = 0,$$

for an integer k, the value of which should be stated. **(5 marks)**

(Total 11 marks)

5. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. **(2)**

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **(3)**

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **(4)**

(d) **Hence** solve the equation

$$\cos 5\theta + \cos 3\theta + \cos\theta = 0, \quad \text{where } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$ **(6)**

(Total 15 marks)

6. The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$. **(4)**

(b) Use the binomial theorem to expand $\left(z - \dfrac{1}{z}\right)^5$ giving your answer in simplified form. **(3)**

(c) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. **(5)**

(d) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \,\mathrm{d}\theta$. **(4)**

(Total 16 marks)

7. (a) By expressing $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^2\theta\sin^4\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$ **(8)**

(b) **Hence** find the exact value of

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta d\theta.$$ **(4)**

(Total 12 marks)

8. Consider $\omega = \cos\left(\dfrac{2\pi}{3}\right) + \mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(4)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \dfrac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(8)

(Total 16 marks)

9. The integrals C and S are defined by

$$C = \int_0^{\frac{\pi}{2}} e^{2x}\cos 3x dx \qquad \text{and} \qquad S = \int_0^{\frac{\pi}{2}} e^{2x}\sin 3x dx.$$

By considering $C + \mathrm{i}S$ as a single integral, show that

$$C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$$

and obtain a similar expression for S.

[You may assume that the standard result for $\int \mathrm{e}^{kx}\,\mathrm{d}x$ remains true when k is a complex constant, so that $\int \mathrm{e}^{(a+\mathrm{i}b)x}\,\mathrm{d}x = \dfrac{1}{a+\mathrm{i}b}\mathrm{e}^{(a+\mathrm{i}b)x} + \text{constant}$.]

(Total 12 marks)