CONIC SECTIONS

CIRCLE $x^2+y^2=a^2$ $(x-x_0)^2+(y-y_0)^2=a^2$

Tangent at (x_1,y_1) $xx_1+yy_1=a^2$ $(x-x_0)(x_1-x_0)+(y-y_0)(y_1-y_0)=a^2$

ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$

Tangent at (x_1,y_1) $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ $\frac{(x-x_0)(x_1-x_0)}{a^2}+\frac{(y-y_0)(y_1-y_0)}{b^2}=1$

HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$

Tangent at (x_1,y_1) $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ (Similarly)

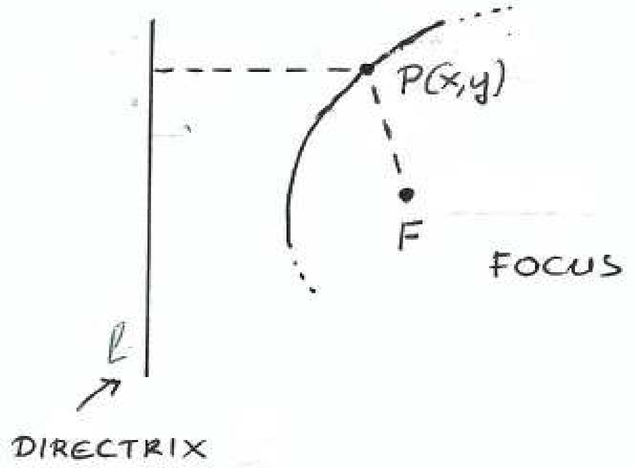
PARABOLA $y^2=4ax$ $(y-y_0)^2=4a(x-x_0)$

Tangent at (x_1,y_1) $yy_1=2a(x+x_1)$ $(y-y_0)(y_1-y_0)=2a(x+x_1-2x_0)$

NOTICE:

- $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ is also a hyperbola
- $x^2=4ay$ is also a parabola
- If $a=b$ in hyperbola: $x^2-y^2=a^2$ RECTANGULAR HYPERBOLA

A. FOCUS - DIRECTRIX DEFINITIONS



Locus of points where $\dfrac{d_{PF}}{d_{P\ell}} = e > 0$ (e constant)

If $e=1$, (i.e. $d_{PF} = d_{P\ell}$) PARABOLA

If $e<1$ (i.e $d_{PF} < d_{P\ell}$) ELLIPSE

If $e>1$ (i.e $d_{PF} > d_{P\ell}$) HYPERBOLA

e is called ECCENTRICITY

NOTICE

When $e \to 0$ then LOCUS $\to$ POINT F (focus)

When $e \to +\infty$ then LOCUS $\to$ LINE ℓ (directrix)

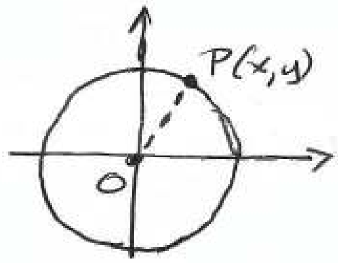
B. STANDARD FORMS OF CONIC SECTIONS

- THE CIRCLE $\boxed{x^2+y^2=a^2}$

CENTER: $O(0,0)$

Locus of points $P(x,y)$

s.t $d_{PO}=a$ (constant)



$$\sqrt{(x-0)^2+(y-0)^2}=a \Rightarrow x^2+y^2=a^2$$

NOTICE: For a different CENTER $C(x_0,y_0)$

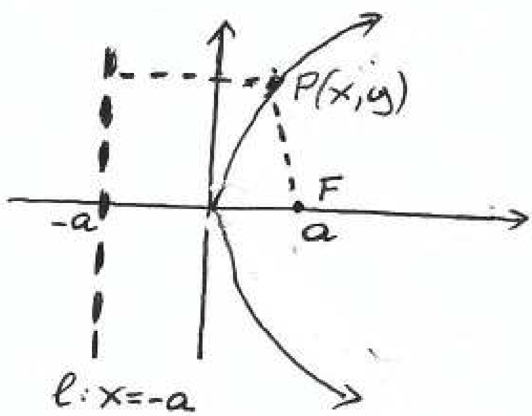
EITHER locus of $P(x,y)$ s.t $d_{PC}=a$

OR translation of $x^2+y^2=a$ by $\binom{x_0}{y_0}$

- THE PARABOLA $\boxed{y^2=4ax}$

FOCUS $F(a,0)$

DIRECTRIX $\ell: x=-a$



Locus of $P(x,y)$ s.t.

$$d_{PF}=d_{P\ell} \Rightarrow \sqrt{(x-a)^2+y^2}=x+a$$

$$\Rightarrow (x-a)^2+y^2=(x+a)^2$$

$$\Rightarrow x^2-2ax+a^2+y^2=x^2+2ax+a^2$$

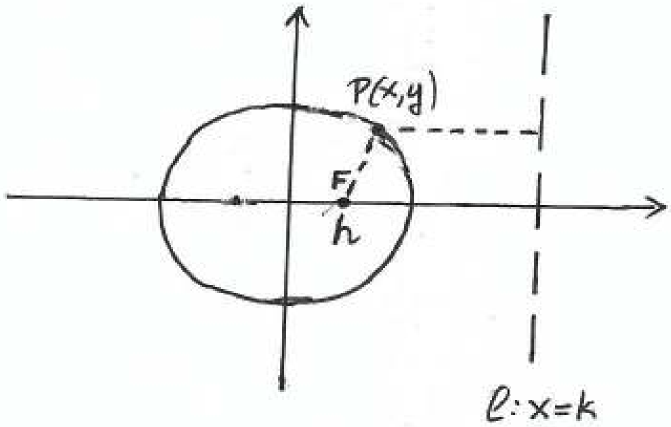
$$\Rightarrow y^2=4ax$$

▶ THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

METHOD A:

FOCUS $F(h,0)$

DIRECTRIX $\ell : x=k$



Locus of points $P(x,y)$

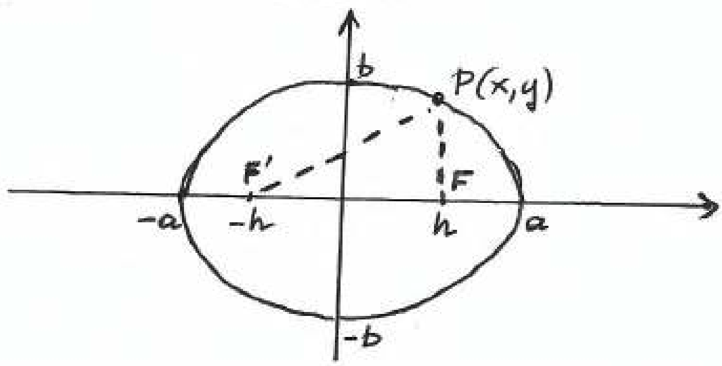
s.t. $\frac{d_{PF}}{d_{P\ell}}=e<1$

METHOD B:

TWO FOCI $F'(-h,0)$ and $F(h,0)$

Locus of points $P(x,y)$ s.t.

$$d_{PF}+d_{PF'}=2a \quad \text{(constant sum)}$$



Both methods result to an equation

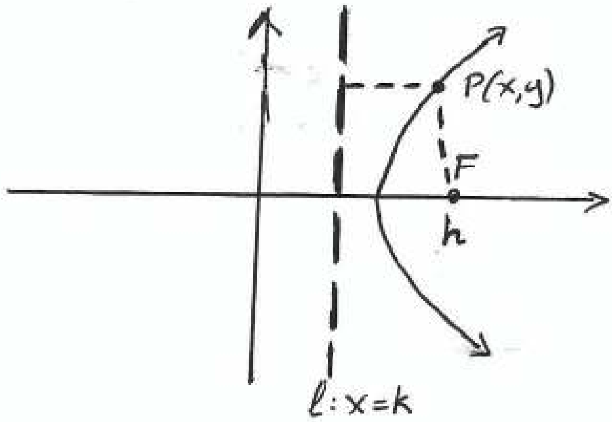
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with x-intercepts $x=\pm a$, y-intercepts $y=\pm b$

▶ THE HYPERBOLA $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$

METHOD A:

Focus $F(h,0)$

DIRECTRIX $\ell : x = k$



Locus of points $P(x,y)$

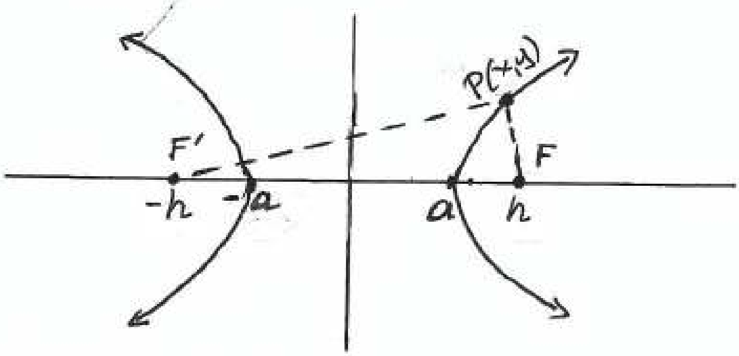
s.t. $\dfrac{d_{PF}}{d_{P\ell}} = e > 1$

METHOD B:

Two FOCI $F'(-h,0)$ and $F(h,0)$

Locus of points $P(x,y)$ s.t

$|d_{PF} - d_{PF'}| = 2a$ (constant difference)

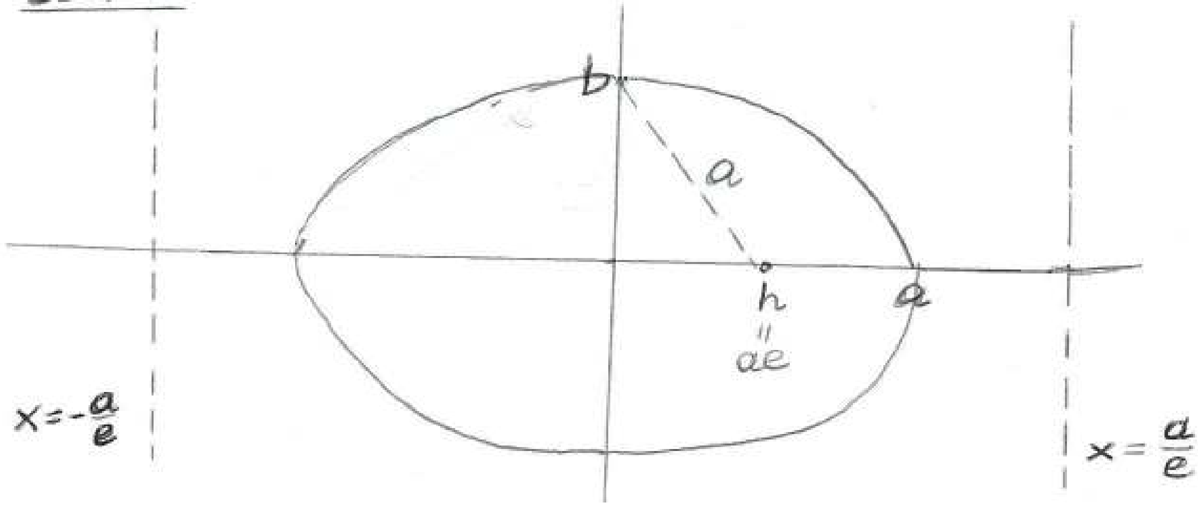


Both methods result to an equation:

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with x-intercepts $x = \pm a$

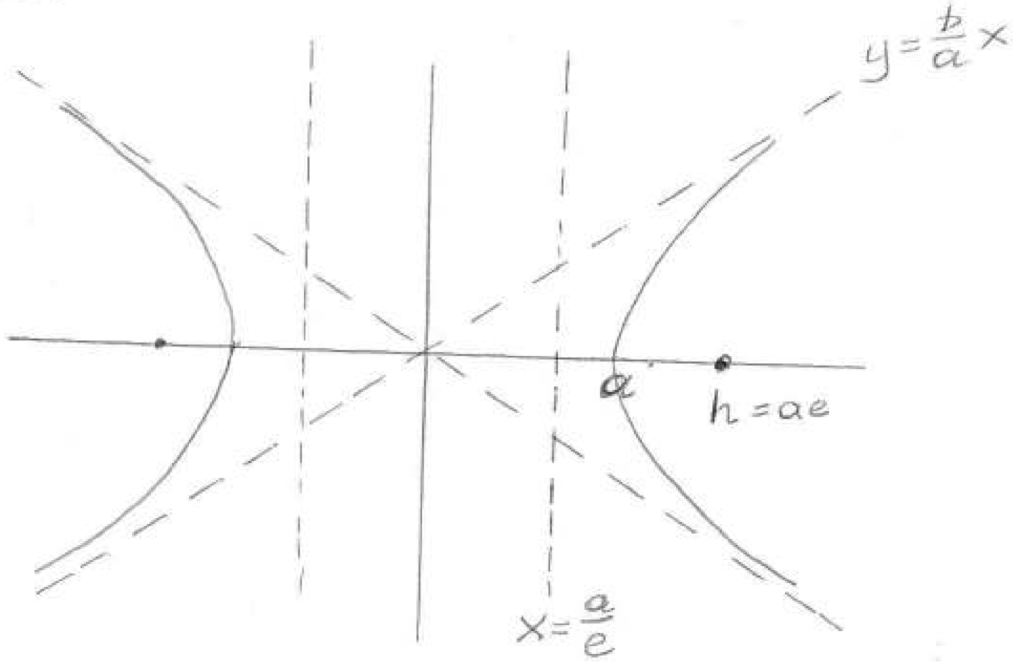
C. RELATIONS BETWEEN FOCI AND DIRECTRIX

ELLIPSE



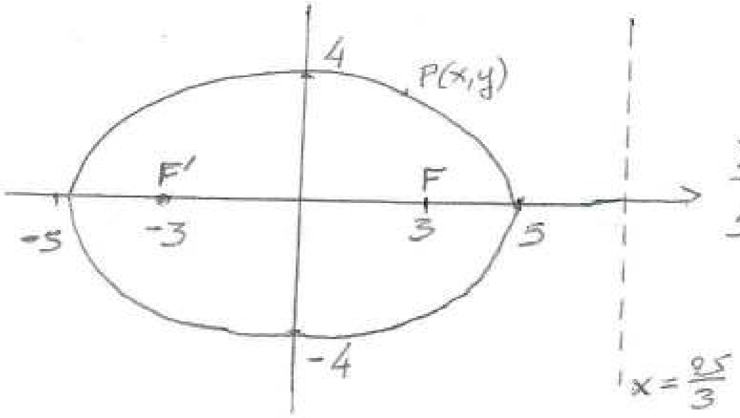
$$h^2+b^2=a^2 \quad , \quad e=\frac{h}{a}$$

HYPERBOLA



$$a^2+b^2=h^2 \qquad e=\frac{h}{a}$$

D. EXAMPLE OF AN ELLIPSE



$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$$

METHOD A : Given two foci $F(3,0)$, $F'(-3,0)$
Find the locus of the points $P(x,y)$ s.t.

$$d_{PF} + d_{PF'} = 10 \qquad (a=5 \Rightarrow 2a=10)$$

We obtain

$$\sqrt{(x+3)^2+y^2} + \sqrt{(x-3)^2+y^2} = 10 \qquad (1)$$

Let $$\sqrt{(x+3)^2+y^2} - \sqrt{(x-3)^2+y^2} = A \qquad (2)$$

(conjugate)

Then

(1) × (2) : $[(x+3)^2+y^2] - [(x-3)^2+y^2] = 10A$

$$\Rightarrow (x+3)^2-(x-3)^2 = 10A$$

$$\Rightarrow 12x = 10A \Rightarrow A = \frac{6x}{5}$$

(1) + (2) : $2\sqrt{(x+3)^2+y^2} = 10+A \Rightarrow 2\sqrt{x^2+6x+9+y^2} = 10+\frac{6x}{5}$

$$\Rightarrow \sqrt{x^2+y^2+6x+9} = 5+\frac{3x}{5}$$

$$\Rightarrow x^2+y^2+6x+9 = 25+6x+\frac{9x^2}{25}$$

$$\Rightarrow 25x^2+25y^2+150x+225 = 625+150x+9x^2$$

$$\Rightarrow 16x^2+25y^2 = 400 \Rightarrow \boxed{\frac{x^2}{25}+\frac{y^2}{16}=1}$$

METHOD B : Given Focus $F(3,0)$, Directrix $\ell: x = \frac{25}{3}$

eccentricity $e = \frac{3}{5} < 1$

Find the locus of the points $P(x,y)$ s.t.

$$\frac{d_{PF}}{d_{P\ell}} = \frac{3}{5}$$

We obtain:

$$\frac{\sqrt{(x-3)^2+y^2}}{\frac{25}{3}-x} = \frac{3}{5} \Rightarrow \sqrt{x^2-6x+9+y^2} = 5 - \frac{3x}{5}$$

$$\Rightarrow x^2+y^2-6x+9 = 25 - 6x + \frac{9x^2}{25}$$

$$\Rightarrow x^2+y^2 = 16 + \frac{9x^2}{25}$$

$$\Rightarrow 25x^2+25y^2 = 400+9x^2$$

$$\Rightarrow 16x^2+25y^2 = 400$$

$$\Rightarrow \boxed{\frac{x^2}{25} + \frac{y^2}{16} = 1}$$

NOTICE

- Given $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (i.e. $a=5$, $b=4$)

  we can find foci: $h^2 = a^2 - b^2 \Rightarrow h = 3$ $F(3,0)$ $F'(-3,0)$

  eccentricity: $e = \frac{h}{a} = \frac{3}{5}$ directrix $x = \frac{a}{e} = \frac{25}{3}$

- Given focus $F(3,0)$, directrix $x = \frac{25}{3}$ and $e = \frac{3}{5}$

  we can find a, b: $h = ae \Rightarrow a = 5$ $b^2 = a^2 - h^2 = 16 \Rightarrow b = 4$

  Thus $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$

E. GENERAL FORM: $ax^2+by^2+cx+dy+e=0$

► If $a\neq 0, b\neq 0$

We can complete squares for x and y:

$$a(x-x_0)^2+b(y-y_0)^2=F$$

Let $F\neq 0$

- If $a=b$ CIRCLE (or EMPTY SET)
- If $ab>0$ ELLIPSE (or EMPTY SET)
- If $ab<0$ HYPERBOLA

NOTICE: If $F=0$ we obtain a POINT or TWO LINES

e.g. $(x-1)^2+(y-2)^2=0 \Rightarrow (x,y)=(1,2)$

$(x-1)^2-(y-2)^2=0 \Rightarrow y=x+1$ or $y=-x+3$

► If $a=0, b\neq 0$

We can complete square for y

$$(y-y_0)^2=-\frac{c}{b}x+F$$

- If $c\neq 0$ PARABOLA $(y-y_0)^2=A(x-x_0)$
- If $c=0$ EMPTY SET or ONE LINE or TWO LINES

NOTICE: If $a\neq 0, b=0$ similar results obtained.

EXAMPLES

1. $x^2+y^2-2x-4y+4=0$

   $\Rightarrow (x-1)^2+(y-2)^2=1$ CIRCLE

2. $x^2+y^2-2x-4y+5=0$

   $\Rightarrow (x-1)^2+(y-2)^2=0$ POINT $(x,y)=(1,2)$

3. $x^2+y^2-2x-4y+6=0$

   $\Rightarrow (x-1)^2+(y-2)^2=-1$ EMPTY SET

4. $x^2+2y^2-2x-4y+2=0$

   $\Rightarrow (x-1)^2+2(y-1)^2=1$ ELLIPSE

5. $x^2+2y^2-2x-4y+3=0$

   $\Rightarrow (x-1)^2+2(y-1)^2=0$ POINT $(x,y)=(1,1)$

6. $x^2+2y^2-2x-4y+4=0$

   $\Rightarrow (x-1)^2+2(y-1)^2=-1$ EMPTY SET

7. $x^2-2y^2-2x+4y-2=0$

   $\Rightarrow (x-1)^2-2(y-1)^2=1$ HYPERBOLA

8. $x^2-y^2-2x+4y-3=0$

   $\Rightarrow (x-1)^2-(y-2)^2=0 \Rightarrow y-2=\pm(x-1)$ TWO LINES

9. $y^2-4x-2y+9=0$

   $\Rightarrow (y-1)^2=4(x-2)$ PARABOLA

F. MORE GENERAL FORM: $ax^2+2bxy+cy^2+dx+ey+f=0$

The extra term is $\underline{2bxy}$

$ax^2+2bxy+cy^2$ is equal to $(x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Our wish is to eliminate the xy-term:

$a'x^2+c'y^2$ which corresponds to $(x\ y)\begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

That is, to diagonalise

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \longrightarrow \begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix}$$

in an appropriate way.

<u>LEMMA</u>

If $P=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2+b^2=1$ then $P^T=P^{-1}$

(easy to verify)

It can be shown that for $A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ we can find $P^{-1}AP=D$ (diagonalisation)

where P is as above. Thus $P^TAP=D$

Then we use the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x' \\ y' \end{pmatrix} \quad \left(\text{in fact } \begin{pmatrix} x' \\ y' \end{pmatrix} = P^T\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

But $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x' \\ y' \end{pmatrix} \overset{T}{\Rightarrow} (x\ y) = (x'\ y')P^T$

Then

$$
\begin{aligned}
ax^2+2bxy+cy^2 &= (x\ y)A\begin{pmatrix} x \\ y \end{pmatrix} \\
&= (x'\ y')P^TAP\begin{pmatrix} x' \\ y' \end{pmatrix} \\
&= (x'\ y')D\begin{pmatrix} x' \\ y' \end{pmatrix} \\
&= (x'\ y')\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(x'\ y') \\
&\Rightarrow \lambda_1 x'^2 + \lambda_2 y'^2
\end{aligned}
$$

<u>Notice</u>

- When you find the first eigenvector $\begin{pmatrix} m \\ n \end{pmatrix}$, it is certain that the second eigenvector, can be $\begin{pmatrix} -n \\ m \end{pmatrix}$.
  Just normalise them, i.e multiply by $\frac{1}{\sqrt{m^2+n^2}}$
  Then $P = \begin{pmatrix} m' & -n' \\ n' & m' \end{pmatrix}$ satisfies $P^T = P^{-1}$
- Since $m'^2+n'^2 = 1$
  $P = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$ for some ϑ
  i.e it is a rotation.

EXAMPLE

$$5x^2+4xy+5y^2=21$$

$$A=\begin{pmatrix}5 & 2\\ 2 & 5\end{pmatrix} \quad \det$$

Eigenvalues: $\begin{vmatrix}5-\lambda & 2\\ 2 & 5-\lambda\end{vmatrix}=0 \Leftrightarrow \lambda^2-10\lambda+21=0 \Leftrightarrow \lambda_1=7, \lambda_2=3$

For $\lambda=7$ $\quad \left.\begin{array}{l}-2x+2y=0\\ 2x+2y=0\end{array}\right\} \Rightarrow y=x \to \begin{pmatrix}x\\ x\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}x$

For $\lambda=3$ $\quad \left.\begin{array}{l}2x+2y=0\\ 2x+2y=0\end{array}\right\} \Rightarrow x=-y \to \begin{pmatrix}x\\ -x\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}x$

We normalise the columns by dividing by

$\sqrt{1^2+1^2}=\sqrt{2}$

$$P=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} \qquad \begin{pmatrix}x\\ y\end{pmatrix}=P\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'\\ \frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\end{pmatrix}$$

i.e. $\quad x=\frac{1}{\sqrt{2}}(x'-y')$

$\quad\quad\;\; y=\frac{1}{\sqrt{2}}(x'+y')$

The original relation becomes

$$5\,\tfrac{1}{2}(x'-y')^2+4\,\tfrac{1}{2}(x'-y')(x'+y')+5\,\tfrac{1}{2}(x'+y')^2=21$$

$$\Leftrightarrow \tfrac{5}{2}(x'^2-2x'y'+y'^2)+2(x'^2-y'^2)+\tfrac{5}{2}(x'^2+2x'y'+y'^2)=21$$

$$\Leftrightarrow 7x'^2+3y'^2=21 \Leftrightarrow \frac{(x')^2}{3}+\frac{(y')^2}{7}=1$$

We can also find the rotation we applied

The transformation matrix is

$$P^{-1} = P^{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

So $\vartheta = 45°$

Therefore if we apply an anticlockwise rotation of $45°$ in the original relation we obtain the ellipse $\frac{x^2}{3} + \frac{y^2}{7} = 1$

EXAMPLE

$$5x^2 + 4xy + 5y^2 - \sqrt{2}x - 13\sqrt{2}y = 2$$

The same transformation as above

$x = \frac{1}{\sqrt{2}}(x' - y')$

$y = \frac{1}{\sqrt{2}}(x' + y')$ gives

$5x^2 + 4xy + 5y^2 \rightarrow 7x'^2 + 3y'^2$ (as above)

$-\sqrt{2}x - 13\sqrt{2}y \rightarrow -\sqrt{2}\frac{1}{\sqrt{2}}(x' - y') - 13\sqrt{2}\frac{1}{\sqrt{2}}(x' + y') = -14x' - 12y'$

Thus

$$7x'^2 + 3y'^2 - 14x' - 12y' = 2$$

The new equation

$$7x^2+3y^2-14x-12y=2$$

represents an ellipse; Complete squares:

$$7(x^2-2x+1)-7+3(y^2-4y+4)-12=2.$$

$$7(x-1)^2+3(y-2)^2=21$$

$$\frac{(x-1)^2}{3}+\frac{(y-2)}{7}=21 \qquad \text{center } (1,2)$$

Question: What is the center of the original ellipse?

$$\begin{pmatrix}1\\2\end{pmatrix}=P^T\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix}=P\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-1/\sqrt{2}\\ 3/\sqrt{2}\end{pmatrix}$$

Thus the center was $(-1/\sqrt{2}, 3/\sqrt{2})$