HAEF IB - MATH HL

TEST 7

STATISTICS AND PROBABILITY

by Christos Nikolaidis

Marks: ____ **/80**

Grade: ____ **%**

Name: SOLUTIONS

Date: **8 – 11 – 2016**

Questions

1. *[Maximum mark: 6]*

A Math HL class consists of 11 students, four named Alex and seven with completely different names.

(a) Find the number of ways the 11 students can sit in a row if all four students named Alex sit next to each other. *[2 marks]*

(b) The students write their first name on a card. Find the number of ways that the 11 cards can be put in a row *[2 marks]*

(c) A 5-member committee has to be chosen at random from the class. Find the probability that the students named Alex constitute the majority in this committee. *[2 marks]*

2. *[Maximum mark: 4]*

Consider the grades of 11 students in a math test. The mean is 6.36 (in 3sf), the median is 7, the range is 2 and the interquartile range is 1. Find the eleven grades.

3. *[Maximum mark: 6]*

In an IB School, 30% of the Math HL students also attend Economics SL, while 40% of the Economics SL students also attend Math HL. It is also known that 42% of the students attend neither Math HL nor Economics SL. Show that the events "Math HL student" and "Economics SL student" are independent.

4. *[Maximum mark: 5]*

A Math HL class consists of 11 students. Two of them attend Further Math as well. The probability that a Further Math student is absent in a Math HL class is 8% while this probability for each of the remaining students falls to 5%. Given that a student is absent in a math HL class on Tuesday morning, find the probability that the student is not a Further Math student.

5. *[Maximum mark: 6]*

Nikos spent a whole day with his 3D printer to produce six discs, identical in all respects except color, namely **1 white**, **2 black** and **3 red** discs,. Manos showed a preference for the **black** discs while Dimitra for the **red** ones. Then Nikos suggested the following game to Manos and Dimitra: "Manos, you first, pick a disc at random. If it is your favorite color you get it, otherwise place it back. Then, Dimitra, the same applies to you. Carry on until one of you wins their favorite color".

(a) Find the probability that Manos wins in his second trial. *[2 marks]*

(b) Find the probability that Manos wins the game and deduce whether Nikos's game is fair. *[4 marks]*

6. *[Maximum mark: 10]*

The time taken for Alex Hatz to come to school is normally distributed with mean 30 min and standard deviation 7 min.

(a) Given that Alex leaves his home at 8:05, find the probability that he will be late at school (i.e. he arrives after 8:30, when lessons start!) *[2 marks]*

(b) If Alex wishes to be on time (i.e. before 8:30) at least 90% of the days. Find the time he should leave home every day. *[2 marks]*

The time taken for another student, called Ioannis, to come to school is also normally distributed with mean 30 min and standard deviation σ.

(c) Given that Ioannis leaves his home at 8:02, and is late 60% of the morning classes, find the standard deviation σ. *[3 marks]*

(d) For a particular 5-days week, Ioannis leaves his home at 7:59. Find the probability that he is on time at least twice. *[3 marks]*

7. *[Maximum mark: 7]*

Consider again that legendary Math HL class consisting of 11 students with four them named Alex. The math teacher suddenly forgot the names of the students and decided to select students at random until he finds a student whose name **is not** Alex. Let the variable X denote the number of students selected.

(a) Write down the possible values of X. *[1 mark]*

(b) Find $E(X)$. *[4 marks]*

(c) Find $Var(X)$. *[2 marks]*

8. *[Maximum mark: 6]*

Dimitris is repeatedly throwing a plastic bottle of water on his desk with a hope to see the bottle standing upright. But a simple statistical analysis shows that he **fails nine out of ten** times.

(a) Find the probability that in his following 20 attempts he will **fail** at most 18 times. *[3 marks]*

(b) Find the least number of trials needed so that the probability to **succeed** at least 3 times is more than 0.2 *[3 marks]*

9. *[Maximum mark: 10]*

In the official IB documents the word "*reflection*" appears **once every two pages** on average! You may assume that the frequency of this word follows a Poisson distribution.

(a) Find the probability that a three page IB document contains the word "*reflection*" at least once. *[2 marks]*

(b) Given that the word "*reflection*" appears at least once in a three page IB document, find the probability that it appears at most 5 times. *[2 marks]*

(c) The probability that an IB CAS brochure contains the word "*reflection*" exactly 10 times is 0.1104 (to 4dp). Find the number of pages of the brochure. *[3 marks]*

(d) If you are given eight different pages of an IB document find the probability that exactly three of them do not contain the word "*reflection*". *[3 marks]*

10. *[Maximum mark: 11]*

The time T (in minutes) that Philip can remain silent (not even wispering!) during a Math HL course is random with probability density function given by

$$f(t) = \begin{cases} 0.2e^{-0.2t}, & t \geq 0 \\ 0 & otherwise \end{cases}$$

(a) Find $\int_0^b f(t)dt$ and explain why f can be a pdf. *[3 marks]*

(b) Find the probability that Philip will remain silent for at least two minutes. *[2 marks]*

(c) Write down an expression for $E(T)$; use your GDC to estimate its value. *[3 marks]*

(d) Find the exact value of the median of T in the form $\ln a$, where $a \in Z$ *[3 marks]*

11. *[Maximum mark: 9]*

A Math HL class consists of n students. The mean of their scores in a math test is 75%. A friend of Petros from another section of the school called "Eniaio" asks him about his score but Petros does not reveal it. He simply says:

"If you remove my score, which is a %, the new mean will be 73%"

(a) Show that $a - 2n = 73$. *[3 marks]*

But Petros, who likes mathematical puzzles, continues:

"the sum of the two digits of my score is 14 but remember that math HL classes have more than 4 students"

(b) Find Petros's score and the size of the class. *[2 marks]*

(c) The variance of the scores is 78.7. Show that the new variance, if you remove Petros's score, is 42.6 (approximately). *[4 marks]*

Disclaimer:

The characters and events in this paper are fictitious. Any apparent similarity to real persons is not intended by the author and is either a coincidence or the product of your own imagination."

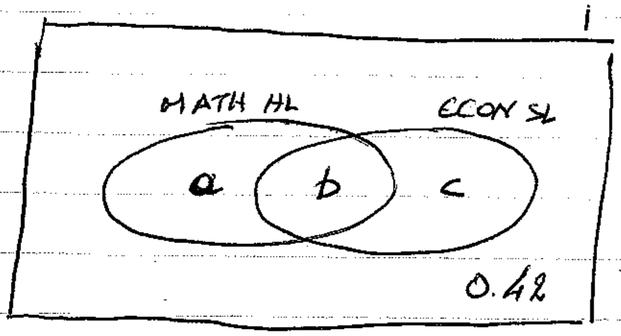
1. (a) $8! \cdot 4!$ (b) $\frac{11!}{4!}$

(c) $$\frac{\binom{4}{3}\binom{7}{2}+\binom{4}{4}\binom{7}{1}}{\binom{11}{5}} = \frac{4\cdot 21+7}{\frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5}} = \frac{13}{66}$$

2. $5, 5, 6, 6, 6, 7, 7, 7, 7, 7, 7$

Q_1 Q_2 Q_3

3.



$$a+b+c=0.58 \quad (*)$$

$$P(ECON/MATH)=0.30 \Rightarrow \frac{b}{a+b}=0.3 \Rightarrow b=0.3a+0.3b$$
$$\Rightarrow a=\frac{7}{3}b$$

$$P(MATH/ECON)=0.40 \Rightarrow \frac{b}{b+c}=0.4 \Rightarrow b=0.4b+0.4c$$
$$\Rightarrow c=\frac{3}{2}b$$

Hence by $(*)$ $\frac{7}{3}b+\frac{3}{2}b+b=0.58 \Rightarrow b=0.12$

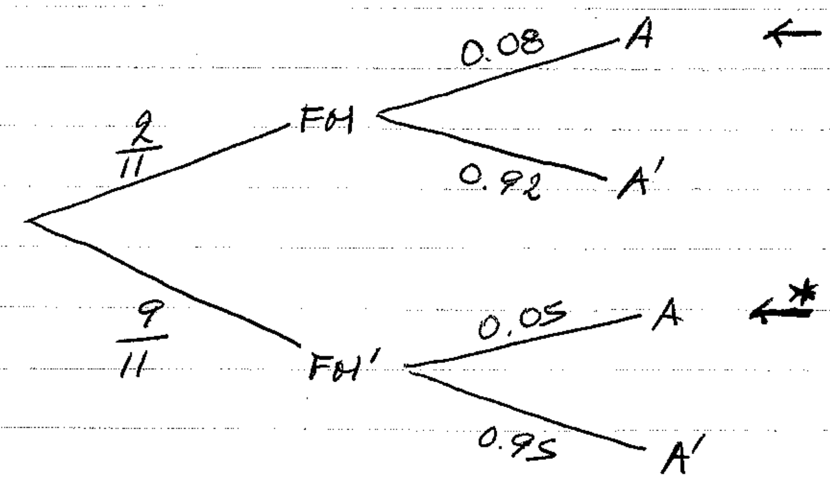
So $a=0.28$

$c=0.18$

$P(MATH \cap ECON)=0.12$ $P(MATH)\cdot P(ECON)=0.4\times 0.3=0.12$

Hence the events are independent.

4.



$$P(FH' \mid A) = \frac{\frac{9}{11} \times 0.05}{\frac{2}{11} \times 0.08 + \frac{9}{11} \times 0.05} = \frac{45}{61} \quad (\cong 0.738)$$

5. For Manos success $\frac{1}{3}$, failure $\frac{2}{3}$

For Dimitra success $\frac{1}{2}$, failure $\frac{1}{2}$

(a) $P(\text{Manos in } 2^{nd}) = \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{9}$

(b) $P(\text{Manos wins}) = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} + \left(\frac{2}{3} \cdot \frac{1}{2}\right)^2 \cdot \frac{1}{3} + \ldots$

G.S with $u_1 = \frac{1}{3}$ $r = \frac{1}{3}$

$$S_\infty = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

Hence the probability for both is $\frac{1}{2}$ and the game is fair.

6. $\mu = 30 \quad \sigma = 7$

(a) $P(X > 0.25) = 0.762$ (Ncd)

(b) $P(X < a) = 0.90 \Rightarrow a \doteq 39$ (InvN)

Hence Alex should leave at 7:51

Now $\mu = 30 \quad \sigma = ?$

(c) $P(X' > 28) = 0.6$

$$Z_{GDC} = \frac{28-30}{\sigma} \Rightarrow \frac{-2}{\sigma} = -0.253 \Rightarrow \sigma \cong 7.91$$

(d) $P(X' < 31) = 0.55$

Now $B(n,p)$ with $n=5 \quad p = 0.55$

$P(Y \geq 2) = 0.869$

7 (a), (b)

X	1	2	3	4	5
$P(X=x)$	$\frac{7}{11}$	$\frac{14}{55}$	$\frac{14}{165}$	$\frac{7}{330}$	$\frac{1}{330}$

$P(X=2) = \frac{4}{11} \cdot \frac{7}{10} = \frac{14}{55}$ $\qquad P(X=3) = \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{7}{9} = \frac{14}{165}$

$P(X=4) = \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9} \frac{7}{8} = \frac{7}{330}$ $\qquad P(X=5) = 1 - \text{all} = \frac{1}{330}$

$$E(X) = 1 \times \frac{1}{7} + 2 \times \frac{14}{55} + 3 \times \frac{14}{165} + 4 \times \frac{7}{330} + 5 \times \frac{1}{330} = \mathbf{1.5}$$

(c) $E(X^2) = \frac{17}{6}$ $\qquad Var(X) = \frac{17}{6} - \left(\frac{3}{2}\right)^2 = \frac{7}{12} \ (= 0.583)$

8. (a) $B(20, \frac{9}{10})$ $\quad P(X \leq 18) = 0.608$

(b) $B(n, \frac{1}{10})$ $\quad P(Y \geq 3) > 0.2 \Rightarrow P(Y \leq 2) < 0.8$

$$0.9^n + n\,0.1 \times 0.9^{n-1} + \frac{n(n-1)}{2}\,0.1^2 \cdot 0.9^{n-2} < 0.8$$

By GDC $\quad n > 15.6 \quad$ Hence $n = 16$

9. frequency $= 0.5$ per page $\quad Po(\mu)$

(a) $Po(1.5)$ $\quad P(X \geq 1) = 0.777$

(b) $Po(1.5)$ $\quad P(X \leq 5 / X \geq 1) = \dfrac{P(1 \leq X \leq 5)}{P(X \geq 1)} = 0.994$

(c) $Po(\mu)$ $\quad P(X = 10) = 0.1104 \Rightarrow \dfrac{e^{-\mu}\mu^{10}}{10!} = 0.1104$

by GDC $\quad \mu \cong 8.5 \quad$ or $\quad \mu = \cdots$

but only $\frac{8.5}{0.5} \cong 17$ gives an integer

Hence 17 pages

(d) $Po(0.5)$ $\quad P(X = 0) = 0.607$

Then $B(8, 0.607)$ $\quad P(Y = 3) = 0.117$

10. (a) $\int_0^b 0.2e^{-0.2t}dt = \left[-e^{-0.2t}\right]_0^b = 1 - e^{-0.2b}$

$f(t) \geq 0$ and the integral $\to 1$
when $b \to +\infty$

(b) $P(T \geq 2) = 1 - P(T \leq 2)$

$= 1 - (1 - e^{-0.4}) = e^{-0.4}$ $(= 0.670)$

(c) $E(T) = \int_0^{+\infty} 0.2t e^{-0.2t}\, dt = 5$ (by GDC)

(for $+\infty$ we use a large number)

(d) $\int_0^{M} 0.2e^{-0.2t}dt = \frac{1}{2} \Leftrightarrow \left[-e^{-0.2t}\right]_0^{M} = \frac{1}{2}$

$\Leftrightarrow 1 - e^{-0.2M} = \frac{1}{2} \Leftrightarrow e^{-0.2M} = \frac{1}{2} \Leftrightarrow -0.2M = \ln\frac{1}{2}$

$\Leftrightarrow 0.2M = \ln 2 \Leftrightarrow M = 5\ln 2 \Leftrightarrow M = \ln 2^5 = \ln 32$

11. $\frac{\Sigma x_i}{n} = 75 \Rightarrow \Sigma x_i = 75n$

(a) $\frac{\Sigma x_i - a}{n-1} = 73 \Rightarrow 75n - a = 73(n-1) \Rightarrow a - 2n = 73$

(b) If $a = 95$ $n = 11$ ✓
If $a = 86$ $n = 6.5$ rejected
If $a = 77$ $n = 2$ rejected

(c) $\frac{\Sigma x_i^2}{11} - 75^2 = 78.7 \Rightarrow \Sigma x_i^2 = 62{,}740.7$

NEW VAR $= \frac{\Sigma x_i^2 - 95^2}{10} - 73^2 = 42.57 \cong 42.6$