HAEF IB - MATH HL

TEST 7

COMPLEX NUMBERS

by Christos Nikolaidis

B

Name:____________________________________

Date:________________

Marks:____**/100**

Grade: _____

Questions

1. Solve the following equation for z, where z is a complex number.

$$\frac{z}{3+4\mathrm{i}}+\frac{z-1}{5\mathrm{i}}=\frac{5}{3-4\mathrm{i}}$$

Give your answers in the form $a+b\mathrm{i}$ where $a,b \in Z$.

(Total 6 marks)

2. Find the values of n such that $(1+\sqrt{3}\,\mathrm{i})^n$ is a real number.

(Total 5 marks)

3. Consider the equation $z^3 + az^2 + bz + c = 0$, where $a, b, c \in \mathbb{R}$. The points in the Argand diagram representing the three roots of the equation form the vertices of a triangle whose area is 9. Given that one root is $-1 + 3\mathrm{i}$, find

(a) the other two roots; **(4)**

(b) a, b and c. **(4)**

(Total 8 marks)

4. (a) Solve the equation $z^3 = -2 + 2\mathrm{i}$, giving your answers in modulus–argument form. **(6)**

(b) **Hence** show that one of the solutions is $1 + \mathrm{i}$ when written in Cartesian form. **(2)**

(Total 8 marks)

5. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^5$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. **(4)**

(b) Hence show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **(3)**

(c) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$\cos\frac{\pi}{10}=\sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$. **(8)**

(Total 15 marks)

6. The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$. **(4)**

(b) Use the binomial theorem to expand $\left(z - \dfrac{1}{z}\right)^5$ giving your answer in simplified form. **(3)**

(c) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. **(5)**

(d) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta$. **(4)**

(Total 16 marks)

7. Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$.

(a) Find an expression for z, the common ratio of this series. **(2)**

(b) Write down the value of $|z|$. **(1)**

(c) Write down an expression for the sum to infinity of this series. **(2)**

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$ **(10)**

(Total 15 marks)

8. The roots of the equation $z^3 - 1 = 0$ are defined by 1, ω and ω^2.

(a) Sketch an Argand diagram to show these roots. **(2)**

(b) Show that $1 + \omega + \omega^2 = 0$. **(2)**

(c) **Hence** evaluate

(i) $(2+\omega)(2+\omega^2)$

(ii) $\dfrac{1}{2+\omega} + \dfrac{1}{2+\omega^2}$ **(5)**

(d) Find a cubic equation, with integer coefficients, which has roots

$$2, \quad \frac{1}{2+\omega} \quad \text{and} \quad \frac{1}{2+\omega^2}$$ **(6)**

(Total 15 marks)

9. The integrals C and S are defined by

$$C = \int_0^{\frac{\pi}{2}} e^{2x}\cos 3x\,dx \quad \textbf{and} \quad S = \int_0^{\frac{\pi}{2}} e^{2x}\sin 3x\,dx.$$

By considering $C + \mathrm{i}S$ as a single integral, show that

$$C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$$

and obtain a similar expression for S.

[You may assume that the standard result for $\int e^{kx}\,dx$ *remains true when* k *is a complex constant, so that* $\int e^{(a+ib)x}\,dx = \frac{1}{a+ib}e^{(a+ib)x} + \text{constant}$ *]*

(Total 12 marks)

TEST 7

B

COMPLEX NUMBERS

SOLUTIONS

1) $\frac{z}{3+4i} + \frac{z-1}{5i} = \frac{5}{3-4i} \Leftrightarrow \frac{z(3-4i)}{25} + \frac{(z-1)(-5i)}{25} = \frac{5(3+4i)}{25}$

$\Leftrightarrow 3z - 4iz - 5iz + 5i = 15 + 20i$

$\Leftrightarrow (3-9i)z = 15 + 15i$

$\Leftrightarrow (1-3i)z = 5 + 5i$

$\Leftrightarrow z = \frac{5+5i}{1-3i} = \frac{(5+5i)(1+3i)}{10} = \frac{-10+20i}{10}$

$\Leftrightarrow \boxed{z = -1 + 2i}$

2) $1 + \sqrt{3}\,i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

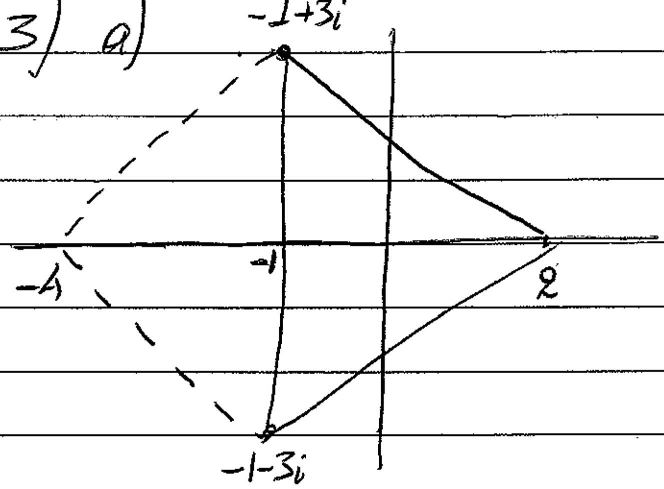
$(1+\sqrt{3}i)^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

This is real if $\sin\frac{n\pi}{3} = 0$

i.e if n is a multiple of 3.

$(n = 0, \pm 3, \pm 6, \ldots)$

3) a)



$A = \frac{bh}{2} \Leftrightarrow 9 = \frac{6 \cdot h}{2} \Leftrightarrow h = 3$

The second root is $-1-3i$

The third root is either 2 or -4

b) if the third root is 2

$$p(z) = (z+1+3i)(z+1-3i)(z-2)$$
$$= [(z+1)^2+9](z-2)$$
$$= (z^2+2z+10)(z-2)$$
$$= z^3+6z-20 \qquad a=0,\ b=6,\ c=-20$$

if the third root is -4

$$p(z) = (z+1+3i)(z+1-3i)(z+4)$$
$$= (z^2+2z+10)(z+4)$$
$$= z^3+6z^2+18z+40 \qquad a=6,\ b=18,\ c=40$$

4) (a) Let $z = r(\cos\vartheta + i\sin\vartheta)$

$$z^3 = r^3(\cos 3\vartheta + i\sin 3\vartheta)$$

For $-2+2i$ modulus $= \sqrt{8}$

argument : $\tan\varphi = -1 \Rightarrow \varphi = \frac{3\pi}{4}$

Hence, $r^3 = \sqrt{8} \Rightarrow r = \sqrt{2}$

$$3\vartheta = \frac{3\pi}{4} + 2k\pi \Rightarrow \vartheta = \frac{3\pi + 8k\pi}{12}$$

$$z_0 = \sqrt{2}\left(\cos\frac{3\pi}{12} + i\sin\frac{3\pi}{12}\right)$$

$$z_1 = \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$z_2 = \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$

(b) $z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\sin\frac{\sqrt{2}}{2}\right) = 1+i$

5) (a) $(\cos\vartheta + i\sin\vartheta)^5 = \cos^5\vartheta + 5\cos^4\vartheta\sin\vartheta\, i - 10\cos^3\vartheta\sin^2\vartheta$
$-10\cos^2\vartheta\sin^3\vartheta\, i + 5\cos\vartheta\sin^4\vartheta + \sin^5\vartheta\, i$

(b) By de Moivre $(\cos\vartheta + i\sin\vartheta)^5 = \cos 5\vartheta + i\sin 5\vartheta$

Consider the real part

$$\cos 5\vartheta = \cos^5\vartheta - 10\cos^3\vartheta\sin^2\vartheta + 5\cos\vartheta\sin^4\vartheta$$

$$= \cos^5\vartheta - 10\cos^3\vartheta(1-\cos^2\vartheta) + 5\cos\vartheta(1-\cos^2\vartheta)^2$$

$$= \cos^5\vartheta - 10\cos^3\vartheta + 10\cos^5\vartheta + 5\cos\vartheta - 10\cos^3\vartheta + 5\cos^5\vartheta$$

$$= 16\cos^5\vartheta - 20\cos^3\vartheta + 5\cos\vartheta$$

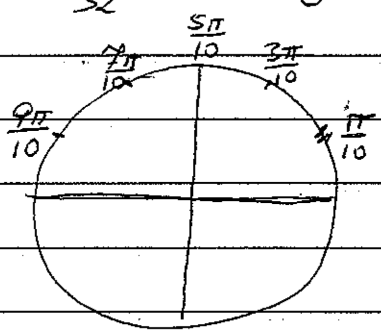
(c) $\cos 5\vartheta = 0 \Leftrightarrow 5\vartheta = \frac{\pi}{2} + k\pi \Leftrightarrow \vartheta = \frac{\pi + 2k\pi}{10}$ $\left(\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \dots\right)$

But $16\cos^5\vartheta - 20\cos^3\vartheta + 5\cos\vartheta = 0$
$\Leftrightarrow \cos\vartheta = 0$ ① or $16\cos^4\vartheta - 20\cos^2\vartheta + 5 = 0$ ②

① $\cos\vartheta = 0 \Leftrightarrow \vartheta = \frac{\pi}{2} + k\pi$.

② $\Delta = 80$ $\quad \cos^2\vartheta = \frac{20 \pm \sqrt{80}}{32} = \frac{20 \pm 4\sqrt{5}}{32} = \frac{5 \pm \sqrt{5}}{8}$

$\cos\vartheta = \pm\sqrt{\frac{5 \pm \sqrt{5}}{8}}$



$$-\sqrt{\frac{5+\sqrt{5}}{8}} < -\sqrt{\frac{5-\sqrt{5}}{8}} < \sqrt{\frac{5-\sqrt{5}}{8}} < \sqrt{\frac{5+\sqrt{5}}{8}}$$

$\downarrow \cos\frac{9\pi}{10}$ $\quad \downarrow \cos\frac{7\pi}{10}$ $\quad \downarrow \cos\frac{3\pi}{10}$ $\quad \downarrow \cos\frac{\pi}{10}$

6) (a) $z^n - z^{-n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$
$= 2i\sin n\theta$

(b) $\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - 10\frac{1}{z} + 5\frac{1}{z^3} - \frac{1}{z^5}$
$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

(c) Consider imaginary parts
$(2\sin\theta)^5 = (2\sin 5\theta) - 5(2\sin 3\theta) + 10(2\sin\theta)$

$\Rightarrow 32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$

$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

(d) $\int_0^{\pi/2} \sin^5\theta\, d\theta = \frac{1}{16}\int_0^{\pi/2} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$
$= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\pi/2}$
$= \frac{1}{16}\left[0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right]$
$= \frac{1}{16} \quad \frac{3 - 25 + 150}{15}$
$= \frac{128}{16 \cdot 15} = \boxed{\frac{8}{15}}$

7) (a) $z = \frac{1}{2}e^{i\theta}$

(b) $|z| = \frac{1}{2}$

(c) $S_\infty = \dfrac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}}$

(d)(i) $S_\infty = \dfrac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)} = \dfrac{2\cos\theta + 2i\sin\theta}{2-\cos\theta - i\sin\theta}$

(ii) $S_\infty = \dfrac{(2\cos\theta + 2i\sin\theta)(2-\cos\theta + i\sin\theta)}{(2-\cos\theta)^2 + \sin^2\theta}$

The real part is

$$\frac{2\cos\theta(2-\cos\theta) - 2\sin^2\theta}{4-4\cos\theta+\cos^2\theta+\sin^2\theta}$$

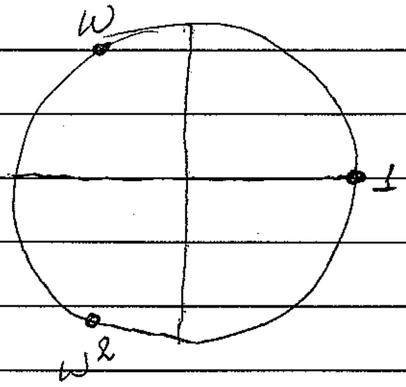
$$= \frac{4\cos\theta - 2\cos^2\theta - 2\sin^2\theta}{5-4\cos\theta}$$

$$= \frac{4\cos\theta - 2}{5-4\cos\theta}$$

But also, the real part of the original series is

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \dots$$

8) (a)



(b) $1+\omega+\omega^2 = \frac{\omega^3-1}{\omega-1} = 0$ since $\omega^3=1$

(c) (i) $(2+\omega)(2+\omega^2) = 4+2\omega^2+2\omega+\omega^3$
$= 4+2(\omega^2+\omega)+1$
$= 5+2(-1)$
$= 3$

(ii) $\frac{1}{2+\omega}+\frac{1}{2+\omega^2} = \frac{2+\omega^2+2+\omega}{(2+\omega)(2+\omega^2)} = \frac{4+\omega+\omega^2}{3} = \frac{4-1}{3} = 1$

(d) $(z-2)\left(z-\frac{1}{2+\omega}\right)\left(z-\frac{1}{2+\omega^2}\right)$

$= (z-2)\left(z^2-\left(\frac{1}{2+\omega}+\frac{1}{2+\omega^2}\right)z+\frac{1}{(2+\omega)(2+\omega^2)}\right)$

$= (z-2)\left(z^2-z+\frac{1}{3}\right)$

The cubic is
$(z-2)(3z^2-3z+1) = 3z^3-3z^2+z-6z^2+6z-2$
$= 3z^3-9z^2+7z-2$

9. $C+iS=\int_0^{\pi/2} e^{2x}(\cos 3x+i\sin 3x)\,dx$

$$=\int_0^{\pi/2} e^{2x} e^{i3x}\,dx$$

$$=\int_0^{\pi/2} e^{2x+i3x}\,dx$$

$$=\int_0^{\pi/2} e^{(2+3i)x}\,dx$$

$$=\frac{1}{2+3i}\left[e^{(2+3i)x}\right]_0^{\pi/2}$$

$$=\frac{1}{2+3i}\left(e^{(2+3i)\pi/2}-1\right)$$

$$=\frac{2-3i}{13}\left(e^{\pi}e^{\frac{3\pi}{2}}-1\right)$$

$$=\frac{2-3i}{13}\left(e^{\pi}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)-1\right)$$

$$=\frac{2-3i}{13}\left(-e^{\pi}i-1\right)$$

$$=-\frac{1}{13}(2-3i)(1+e^{\pi}i)$$

$$=-\frac{1}{13}\left[(2+3e^{\pi})+(2e^{\pi}-3)i\right]$$

Hence $C=-\frac{1}{13}(2+3e^{\pi})$ $\quad S=-\frac{1}{13}(2e^{\pi}-3)$