HAEF IB - MATH HL

TEST 7

COMPLEX NUMBERS

by Christos Nikolaidis

A

Name:______________________________

Date:_______________

Marks:____**/100**

Grade:_____

Questions

1. Given that $\dfrac{z}{z+2} = 2 - \mathrm{i},\ z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

(Total 4 marks)

2. Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that

(a) $\operatorname{Im}\left(z^n + \dfrac{1}{z^n}\right) = 0,\ n \in \mathbb{Z}^+$; **(2)**

(b) $\operatorname{Re}\left(\dfrac{z-1}{z+1}\right) = 0,\ z \neq -1$. **(5)**

(Total 7 marks)

3. Consider the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$.
Given that $1 + \mathrm{i}$ and $1 - 2\mathrm{i}$ are zeros of $p(x)$, find the values of a, b, c and d.

(Total 7 marks)

4.

(*a*) Solve the equation

$$z^3 = (32 + 32\sqrt{3}\mathrm{i}),$$

giving your answers in the form $re^{\mathrm{i}\theta}$ where $r > 0, -\pi < \theta \leqslant \pi$. **(6 marks)**

(*b*) Show that your solutions satisfy the equation

$$z^9 + 2^k = 0,$$

for an integer k, the value of which should be stated. **(5 marks)**

(Total 11 marks)

5. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. **(2)**

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **(3)**

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **(4)**

(d) **Hence** solve the equation

$$\cos 5\theta + \cos 3\theta + \cos\theta = 0, \qquad \text{where } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$ **(6)**

(Total 15 marks)

6. The complex number z is defined as $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin(n\theta)$. **(4)**

(b) Use the binomial theorem to expand $\left(z - \dfrac{1}{z}\right)^5$ giving your answer in simplified form. **(3)**

(c) Hence show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. **(5)**

(d) Find $\int_0^{\frac{\pi}{2}} \sin^5\theta \, \mathrm{d}\theta$. **(4)**

(Total 16 marks)

7. (a) By expressing $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that

$$\cos^2\theta\sin^4\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$ **(8)**

(b) **Hence** find the exact value of

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta d\theta.$$ **(4)**

(Total 12 marks)

8. Consider $\omega = \cos\left(\dfrac{2\pi}{3}\right) + \mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$.

(4)

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \dfrac{\pi}{2}$ on an Argand diagram.

(4)

(c) (i) Expand and simplify $F(z) = (z - 1)(z - \omega)(z - \omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω.

(8)

(Total 16 marks)

9. The integrals C and S are defined by

$$C = \int_0^{\frac{\pi}{2}} e^{2x}\cos 3x dx \qquad \text{and} \qquad S = \int_0^{\frac{\pi}{2}} e^{2x}\sin 3x dx.$$

By considering $C + \mathrm{i}S$ as a single integral, show that

$$C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$$

and obtain a similar expression for S.

[You may assume that the standard result for $\int \mathrm{e}^{kx}\,\mathrm{d}x$ *remains true when* k *is a complex constant, so that* $\int \mathrm{e}^{(a+\mathrm{i}b)x}\,\mathrm{d}x = \dfrac{1}{a+\mathrm{i}b}\mathrm{e}^{(a+\mathrm{i}b)x} + \text{constant}$ *]*

(Total 12 marks)

TEST 7

A

COMPLEX NUMBERS

SOLUTIONS

1) $\frac{z}{z+2} = 2 - i \iff z = 2z - iz + 4 - 2i$

$\iff -z + iz = 4 - 2i$

$\iff (-1+i)z = 4 - 2i$

$\iff z = \frac{4-2i}{-1+i}$

$\iff z = \frac{(4-2i)(-1-i)}{2}$

$\iff z = \frac{-6-2i}{2}$

$\iff \boxed{z = -3 - i}$

2) (a) $z^n + z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta \in \mathbb{R}$

Hence, $\operatorname{Im}(z^n + z^{-n}) = 0$.

(b) $\frac{z-1}{z+1} = \frac{(\cos\theta - 1) + i\sin\theta}{(\cos\theta + 1) + i\sin\theta} \times [(\cos\theta - 1) - i\sin\theta]$

The numerator of $\operatorname{Re}\left(\frac{z-1}{z+1}\right)$ is

$(\cos\theta - 1)(\cos\theta + 1) + \sin^2\theta = \cos^2\theta - 1 + \sin^2\theta = 0$

Hence, $\operatorname{Re}\left(\frac{z-1}{z+1}\right) = 0$.

3) Roots $1+i$, $1-i$, $1-2i$, $1+2i$

$$p(x)=(x-1-i)(x-1+i)(x-1+2i)(x-1-2i)$$

$$=\left[(x-1)^2+1\right]\left[(x-1)^2+4\right]$$

$$=(x^2-2x+2)(x^2-2x+5)$$

$$=x^4-2x^3-2x^3+5x^2+4x^2+2x^2-10x-4x+10$$

$$=x^4-4x^3+11x^2-14x+10$$

Hence $a=-4$ $b=11$ $c=-14$ $d=10$

4) (a) Let $z=re^{i\vartheta}$ $z^3=r^3e^{i3\vartheta}$

For $32+32\sqrt{3}i$ $r=64$ $\tan\vartheta_1=\sqrt{3}$ $\vartheta_1=\frac{\pi}{3}$

Hence $r^3=64 \Leftrightarrow \boxed{r=4}$ $3\vartheta=\frac{\pi}{3}+2k\pi \Leftrightarrow \boxed{\vartheta=\frac{\pi+6k\pi}{9}}$

$z=4e^{i\frac{\pi}{9}}$ $z=4e^{i\frac{7\pi}{9}}$ $z=4e^{\frac{-5\pi}{9}}$

(b) $z^9=\left(4e^{i\frac{\pi+6k\pi}{9}}\right)^9=4^9e^{i(\pi+6k\pi)}=4^9e^{i\pi}$

$$=2^{18}(\cos\pi+i\sin\pi)=-2^{18}$$

Hence $z^9+2^{18}=0$ and $k=18$.

5) (a) $(\cos\vartheta + i\sin\vartheta)^3 = \cos^3\vartheta + 3i\cos^2\vartheta\sin\vartheta - 3\cos\vartheta\sin^2\vartheta - i\sin^3\vartheta$

$= (\cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta) + (3\cos^2\vartheta\sin\vartheta - \sin^3\vartheta)i$

(b) Hence, by considering the real parts

$\cos 3\vartheta = \cos^3\vartheta - 3\cos\vartheta\sin^2\vartheta$

$= \cos^3\vartheta - 3\cos\vartheta(1-\cos^2\vartheta)$

$= 4\cos^3\vartheta - 3\cos\vartheta$

(c) By considering the real part of

$(\cos\vartheta + i\sin\vartheta)^5$

$\cos 5\vartheta = \cos^5\vartheta + 10\cos^3\vartheta\sin^2\vartheta\, i^2 + 5\cos\vartheta\sin^4\vartheta\, i^4$

$= \cos^5\vartheta - 10\cos^3\vartheta(1-\cos^2\vartheta) + 5\cos\vartheta(1-\cos^2\vartheta)^2$

$= \cos^5\vartheta - 10\cos^3\vartheta + 10\cos^5\vartheta + 5\cos\vartheta - 10\cos^3\vartheta + 5\cos^5\vartheta$

$= 11\cos^5\vartheta - 20\cos^3\vartheta + 5\cos\vartheta$

(d) $\cos 5\vartheta + \cos 3\vartheta + \cos\vartheta = 0$

$\Leftrightarrow 16\cos^5\vartheta - 16\cos^3\vartheta + 3\cos\vartheta = 0$

$\Leftrightarrow \cos\vartheta = 0$ ① or $16\cos^4\vartheta - 16\cos^2\vartheta + 3 = 0$ ②

① $\Rightarrow \boxed{\vartheta = \pm\frac{\pi}{2}}$ $\qquad \Delta = 16^2 - 4\cdot 16\cdot 3 = 64$

② $\Rightarrow \cos^2\vartheta = \frac{16 \pm \sqrt{64}}{32} = \frac{16\pm 8}{32} = \frac{2\pm 1}{4}$ $<$ $\frac{1}{4}$, $\frac{3}{4}$

Thus, $\cos\vartheta = \frac{1}{2} \Rightarrow \boxed{\vartheta = \pm\frac{\pi}{3}}$ or $\cos\vartheta = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\vartheta = \pm\frac{\pi}{6}}$

6) (a) $z^n - z^{-n} = \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$
$= 2i\sin n\theta$

(b) $\left(z-\frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - 10\frac{1}{z} + 5\frac{1}{z^3} - \frac{1}{z^5}$
$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

(c) Consider imaginary parts
$(2\sin\theta)^5 = (2\sin 5\theta) - 5(2\sin 3\theta) + 10(2\sin\theta)$

$\Rightarrow 32\sin^5\theta = 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta$

$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

(d) $\int_0^{\pi/2} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\pi/2} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$

$= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\pi/2}$

$= \frac{1}{16}\left[0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right]$

$= \frac{1}{16} \quad \frac{3 - 25 + 150}{15}$

$= \frac{128}{16 \cdot 15} = \boxed{\frac{8}{15}}$

7) (a) $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta \qquad e^{-i\vartheta} = \cos\vartheta - i\sin\vartheta$

$$e^{i\vartheta} + e^{-i\vartheta} = 2\cos\vartheta \Rightarrow \cos\vartheta = \frac{1}{2}\left(e^{i\vartheta} + e^{-i\vartheta}\right)$$

$$e^{i\vartheta} - e^{-i\vartheta} = 2i\sin\vartheta \Rightarrow \sin\vartheta = \frac{1}{2i}\left(e^{i\vartheta} - e^{-i\vartheta}\right)$$

Hence,

$$\cos^2\vartheta\sin^4\vartheta = \frac{1}{4}\left(e^{i\vartheta} + e^{-i\vartheta}\right)^2 \frac{1}{16}\left(e^{i\vartheta} - e^{-i\vartheta}\right)^4$$

$$= \frac{1}{64}\left(e^{2i\vartheta} + 2 + e^{-2i\vartheta}\right)\left(e^{4i\vartheta} - 4e^{2i\vartheta} + 6 - 4e^{-2i\vartheta} + e^{-4i\vartheta}\right)$$

$$= \frac{1}{64}\left(e^{6i\vartheta} - 4e^{4i\vartheta} + 6e^{2i\vartheta} - 4 + e^{-2i\vartheta} + 2e^{4i\vartheta} - 8e^{2i\vartheta} + 12\right.$$

$$\left. - 8e^{-2i\vartheta} + 2e^{-4i\vartheta} + e^{2i\vartheta} - 4 + 6e^{-2i\vartheta} - 4e^{-4i\vartheta} + e^{-6i\vartheta}\right)$$

$$= \frac{1}{64}\left(e^{6i\vartheta} - 2e^{4i\vartheta} - e^{2i\vartheta} + 4 - e^{-2i\vartheta} - 2e^{-4i\vartheta} + e^{-6i\vartheta}\right)$$

$$= \frac{1}{64}\left(2\cos 6\vartheta - 4\cos 4\vartheta - 2\cos 2\vartheta + 4\right)$$

$$= \frac{1}{32}\left(\cos 6\vartheta - 2\cos 4\vartheta - \cos 2\vartheta + 2\right)$$

(b) $$\int_0^{\pi/3} \cos^2\vartheta \sin^4\vartheta \, d\vartheta = \frac{1}{32}\int_0^{\pi/3} \left(\cos 6\vartheta - 2\cos 4\vartheta - \cos 2\vartheta + 2\right) d\vartheta$$

$$= \frac{1}{32}\left[\frac{\sin 6\vartheta}{6} - \frac{\sin 4\vartheta}{2} - \frac{\sin 2\vartheta}{2} + 2\vartheta\right]_0^{\pi/3}$$

$$= \frac{1}{32}\left[\left(0 + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{2\pi}{3}\right) - 0\right] = \frac{\pi}{48}$$

8 (a) (i) $\omega^3 = \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} = \cos 2\pi + i\sin 2\pi = 1$

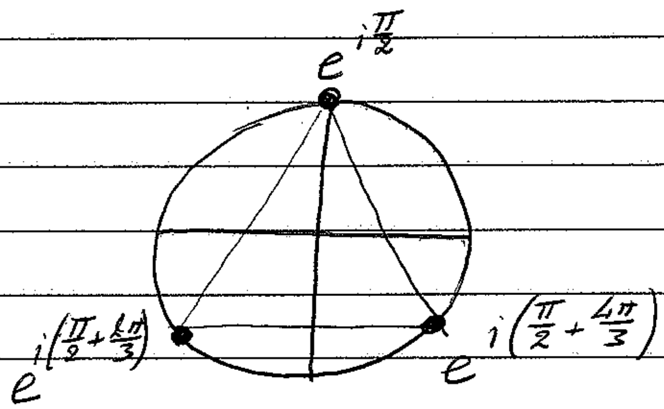
(ii) $1+\omega+\omega^2 = \frac{\omega^3-1}{\omega-1} = 0$

(b) (i) $e^{i\theta} + e^{i\theta}e^{i\frac{2\pi}{3}} + e^{i\theta}e^{i\frac{4\pi}{3}}$

$= e^{i\theta}\left(1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}}\right)$

$= e^{i\theta}(1+\omega+\omega^2) = 0$

(ii)



(c) (i) $F(z) = (z-1)(z-\omega)(z-\omega^2)$

$= (z^2-\omega z - z + \omega)(z-\omega^2)$

$= z^3 - \omega^2 z^2 - \omega z^2 + \omega^3 z - z^2 + \omega^2 z + \omega z - \omega^3$

$= z^3 - (\omega^2+\omega+1)z^2 + (\omega^2+\omega+1)z - \omega^3$

$= z^3 - 1$

(ii) $F(z) = 7 \Leftrightarrow z^3 = 8$

$z_0 = 2$

$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\omega$

$z_2 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2\omega^2$

9.

$$C+iS=\int_0^{\pi/2} e^{2x}(\cos 3x+i\sin 3x)\,dx$$

$$=\int_0^{\pi/2} e^{2x}e^{i3x}\,dx$$

$$=\int_0^{\pi/2} e^{2x+i3x}\,dx$$

$$=\int_0^{\pi/2} e^{(2+3i)x}\,dx$$

$$=\frac{1}{2+3i}\left[\cancel{\int_0^{\pi/2}} e^{(2+3i)x}\right]_0^{\pi/2}$$

$$=\frac{1}{2+3i}\left(e^{(2+3i)\pi/2}-1\right)$$

$$=\frac{2-3i}{13}\left(e^{\pi}e^{i\frac{3\pi}{2}}-1\right)$$

$$=\frac{2-3i}{13}\left(e^{\pi}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)-1\right)$$

$$=\frac{2-3i}{13}\left(-e^{\pi}i-1\right)$$

$$=-\frac{1}{13}(2-3i)(1+e^{\pi}i)$$

$$=-\frac{1}{13}\left[(2+3e^{\pi})+(2e^{\pi}-3)i\right]$$

Hence $C=-\frac{1}{13}(2+3e^{\pi})$ $\quad S=-\frac{1}{13}(2e^{\pi}-3)$