DISCRETE MATHEMATICS

Past Paper Questions

in Number Theory

1. Prove that $3k + 2$ and $5k + 3$, $k \in \mathbb{Z}$ are relatively prime.

(Total 6 marks)

2. (a) Given that the integers m and n are such that $3 \mid (m^2 + n^2)$, prove that $3 \mid m$ and $3 \mid n$.

(7)

(b) **Hence** show that $\sqrt{2}$ is irrational.

(5)

(Total 12 marks)

3. (a) Write 457128 as a product of primes.

(4)

(b) Numbers of the form $F_n = 2^{2^n} + 1$, $n \in \mathbb{N}$ are called Fermat numbers.

Find the smallest value of n for which the corresponding Fermat number has more than a million digits.

(4)

(c) Prove that $22 \mid 5^{11} + 17^{11}$.

(4)

(Total 12 marks)

4. (a) Explaining your method fully, determine whether or not 1189 is a prime number.

(4)

(b) (i) State the fundamental theorem of arithmetic.

(ii) The positive integers M and N have greatest common divisor G and least common multiple L. Show that $GL = MN$.

(6)

(Total 10 marks)

5. (a) Prove that the number 14 641 is the fourth power of an integer in any base greater than 6.

(3)

(b) For $a, b \in \mathbb{Z}$ the relation aRb is defined if and only if $\frac{a}{b} = 2^k, k \in \mathbb{Z}$.

(i) Prove that R is an equivalence relation.

(ii) List the equivalence classes of R on the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(8)

(Total 11 marks)

6. The positive integer N is expressed in base p as $(a_n a_{n-1} \dots a_1 a_0)_p$.

(a) Show that when $p = 2$, N is even if and only if its least significant digit, a_0, is 0.

(5)

(b) Show that when $p = 3$, N is even if and only if the sum of its digits is even.

(6)

(Total 11 marks)

7. Given that $n^2 + 2n + 3 \equiv N \pmod 8$, where $n \in \mathbb{Z}^+$ and $0 \le N \le 7$, prove that N can take one of only three possible values.

(Total 9 marks)

8. (a) Given that $a, b \in \mathbb{N}$ and $c \in \mathbb{Z}^+$, show that if $a \equiv 1 \pmod c$, then $ab \equiv b \pmod c$.

(2)

(b) Using mathematical induction, show that $9^n \equiv 1 \pmod 4$, for $n \in \mathbb{N}$.

(6)

(c) The positive integer M is expressed in base 9. Show that M is divisible by 4 if the sum of its digits is divisible by 4.

(4)

(Total 12 marks)

9. Find the remainder when 67^{101} is divided by 65.

(Total 6 marks)

10. Given that $a, b, c, d \in \mathbb{Z}$, show that

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \equiv 0 \pmod 3.$$

(Total 7 marks)

11. Let a and b be two positive integers.

(a) Show that $\gcd(a,b) \times \operatorname{lcm}(a,b) = ab$

(6)

(b) Show that $\gcd(a, a+b) = \gcd(a, b)$

(7)

(Total 13 marks)

12. (a) Write the number 10 201 in base 8.

(4)

(b) Prove that if a number is divisible by 7 that the sum of its base 8 digits is also divisible by 7.

(5)

(c) Using the result of part (b), show that the number 10 201 is not divisible by 7.

(2)

(Total 11 marks)

13. The positive integer N is expressed in base 9 as $(a_n a_{n-1} \ldots a_0)_9$.

(a) Show that N is divisible by 3 if the least significant digit, a_0, is divisible by 3.

(3)

(b) Show that N is divisible by 2 if the sum of its digits is even.

(3)

(c) Without using a conversion to base 10, determine whether or not $(464860583)_9$ is divisible by 12.

(6)

(Total 12 marks)

14. (a) Define what is meant by the statement $x \equiv y \pmod{n}$ where $x, y, n \in \mathbb{Z}^+$.

(1)

(b) Hence prove that if $x \equiv y \pmod{n}$ then $x^2 \equiv y^2 \pmod{n}$.

(4)

(c) Determine whether or not $x^2 \equiv y^2 \pmod{n}$ implies that $x \equiv y \pmod{n}$.

(4)

(Total 9 marks)

15. (a) Show that a positive integer, written in base 10, is divisible by 9 if the sum of its digits is divisible by 9.

(7)

(b) The representation of the positive integer N in base p is denoted by $(N)_p$.

If $(5^{(126)_7})_7 = (a_n a_{n-1} \ldots a_1 a_0)_7$, find a_0.

(9)

(Total 16 marks)

16. (a) Using Fermat's little theorem, show that, in base 10, the last digit of n is always equal to the last digit of n^5.

(7)

(b) Show that this result is also true in base 30.

(2)

(Total 9 marks)

17. (a) Write down Fermat's little theorem.

(2)

(b) In base 5 the representation of a natural number X is $(k00013(5-k))_5$.

This means that $X = k \times 5^6 + 1 \times 5^2 + 3 \times 5 + (5-k)$.

In base 7 the representation of X is $(a_n a_{n-1} \ldots a_2 a_1 a_0)_7$.

Find a_0.

(5)

(c) Given that $k = 2$, find X in base 7.

(4)

(Total 11 marks)

18. Prove that if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.

(Total 5 marks)

19. (a) Use the Euclidean Algorithm to find the gcd of 324 and 129.

(3)

(b) Hence show that $324x + 129y = 12$ has a solution and find both a particular solution and the general solution.

(6)

(c) Show that there are no integers x and y such that $82x + 140y = 3$.

(2)

(Total 11 marks)

20. (a) Use the Euclidean algorithm to find the greatest common divisor of the numbers 56 and 315.

(4)

(b) (i) Find the general solution to the diophantine equation $56x + 315y = 21$.

(ii) Hence or otherwise find the smallest positive solution to the congruence $315x \equiv 21$ (modulo 56).

(9)

(Total 13 marks)

21. Use the Euclidean Algorithm to find the greatest common divisor of 7854 and 3315.

Hence state the number of solutions to the diophantine equation $7854x + 3315y = 41$ and justify your answer.

(Total 7 marks)

22. (a) Use the Euclidean algorithm to find gcd (12306, 2976).

(5)

(b) Hence give the general solution to the diophantine equation $12306x + 2976y = 996$.

(9)

(Total 14 marks)

23. (a) Given the linear congruence $ax \equiv b(\text{mod } p)$, where $a, b \in \mathbb{Z}$, p is a prime and $\gcd(a, p) = 1$, show that $x \equiv a^{p-2}b \ (\text{mod } p)$.

(4)

(b) (i) Solve $17x \equiv 14 \ (\text{mod } 21)$.

(ii) Use the solution found in part (i) to find the general solution to the Diophantine equation $17x + 21y = 14$.

(10)

(Total 14 marks)

24. (a) (i) Given that $a \equiv d \ (\text{mod } n)$ and $b \equiv c \ (\text{mod } n)$ prove that

$$(a + b) \equiv (c + d) \ (\text{mod } n).$$

(ii) Hence solve the system

$$\begin{cases} 2x + 5y \equiv 1(\text{mod } 6) \\ x + y \equiv 5(\text{mod } 6) \end{cases}$$

(11)

(b) Show that $x^{97} - x + 1 \equiv 0 \ (\text{mod } 97)$ has no solution.

(3)

(Total 14 marks)

25. An arithmetic sequence has first term 2 and common difference 4. Another arithmetic sequence has first term 7 and common difference 5. Find the set of all numbers that are members of both sequences.

(Total 9 marks)

26. (a) Convert the decimal number 51966 to base 16.

(4)

(b) (i) Using the Euclidean algorithm, find the greatest common divisor, d, of 901 and 612.

(ii) Find integers p and q such that $901p + 612q = d$.

(iii) Find the least possible positive integers s and t such that $901s - 612t = 85$.

(10)

(c) In each of the following cases find the solutions, if any, of the given linear congruence.

(i) $9x \equiv 3 \ (\text{mod} 18)$;

(ii) $9x \equiv 3 \ (\text{mod} 15)$.

(5)

(Total 19 marks)

27. Solve the system of linear congruences

$$x \equiv 1(\text{mod } 3);\ x \equiv 2(\text{mod } 5);\ x \equiv 3(\text{mod } 7).$$

(Total 6 marks)

28. (a) Find the general solution for the following system of congruences.

$$N \equiv 3 \pmod{11}$$
$$N \equiv 4 \pmod{9}$$
$$N \equiv 0 \pmod{7}$$

(9)

(b) Find all values of N such that $2000 \le N \le 4000$.

(2)

(Total 11 marks)

29. (a) (i) One version of Fermat's little theorem states that, under certain conditions,

$$a^{p-1} \equiv 1 \pmod{p}.$$

Show that this result is not valid when $a = 4$, $p = 9$ and state which condition is not satisfied.

(ii) Given that $5^{64} \equiv n \pmod{7}$, where $0 \le n \le 6$, find the value of n.

(8)

(b) Find the general solution to the simultaneous congruences

$$x \equiv 3 \pmod{4}$$
$$3x \equiv 2 \pmod{5}.$$

(6)

(Total 14 marks)

30. (a) Given that $ax \equiv b \pmod{p}$ where $a, b, p, x \in \mathbb{Z}^+$, p is prime and a is not a multiple of p, use Fermat's little theorem to show that

$$x \equiv a^{p-2}b \pmod{p}.$$

(3)

(b) Hence solve the simultaneous linear congruences

$$3x \equiv 4 \pmod{5}$$
$$5x \equiv 6 \pmod{7}$$

giving your answer in the form $x \equiv c \pmod{d}$.

(8)

(Total 11 marks)

31. Two mathematicians are planning their wedding celebration and are trying to arrange the seating plan for the guests. The only restriction is that all tables must seat the same number of guests and each table must have more than one guest. There are fewer than 350 guests, but they have forgotten the exact number. However they remember that when they try to seat them with two at each table there is one guest left over. The same happens with tables of 3, 4, 5 and 6 guests. When there were 7 guests per table there were none left over. Find the number of guests.

(Total 10 marks)