MATH HL - OPTION CALCULUS

SOLUTIONS

1. CONTINUITY AND DIFFERENTIABILITY

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A. Practice questions

1. Look at the site!

2.

	Continuous	Differentiable
f	everywhere	not at $x=-2$
h	not at $x=1$	not at $x=1$
p	everywhere	not at $x=\pi/2$

	Continuous	Differentiable
g	everywhere	not at $x=-1$
k	everywhere	not at $x=2$
q	everywhere	everywhere

B. Past paper questions

3.

(a) consider upper or lower limits **M1**

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} 1 = 1\ (= f(0)),\ \lim_{x\to0^+} f(x) = \lim_{x\to0^+} (1-x) = 1\ (= f(0))$ **A1**

$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)$ so f is continuous **AG**

[2 marks]

(b) $\lim_{h\to0^-} \frac{1-1}{h} = 0$ **M1A1**

$\lim_{h\to0^+} \frac{1-h-1}{h} = \lim_{h\to0^+} (-1) = -1$ **A1**

Note: Award ***M1*** for an attempt to find limits in either case.

$\lim_{h\to0^+} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to0^-} \frac{f(0+h)-f(0)}{h}$ so f is not differentiable **AG**

Note: Award ***M1A1A0*** for correct differentiation of left and right sides, ONLY if limits then used to show that both functions have different limits.

[3 marks]

4.

(a) we note that $f(0)=0$, $f(x)=3x$ for $x>0$ and $f(x)=x$ for $x<0$

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} x = 0$ ***M1A1***

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} 3x = 0$ ***A1***

since $f(0)=0$, the function is continuous when $x=0$ ***AG***

$\lim_{h\to0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to0^-} \frac{h}{h} = 1$ ***M1A1***

$\lim_{h\to0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to0^+} \frac{3h}{h} = 3$ ***A1***

these limits are unequal ***R1***

so f is not differentiable when $x=0$ ***AG***

[7 marks]

(b) $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} x\,dx + \int_{0}^{a} 3x\,dx$ ***M1***

$= \left[\frac{x^2}{2}\right]_{-a}^{0} + \left[\frac{3x^2}{2}\right]_{0}^{a}$ ***A1***

$= a^2$ ***A1***

[3 marks]

5.

(a) $\lim\limits_{x\to 1^-} e^{-x^2}\left(-x^3+2x^2+x\right)=\lim\limits_{x\to 1^+}(ax+b)\ \ (=a+b)$ ***M1***

$2e^{-1}=a+b$ ***A1***

differentiability: attempt to differentiate **both** expressions ***M1***

$f'(x)=-2xe^{-x^2}\left(-x^3+2x^2+x\right)+e^{-x^2}\left(-3x^2+4x+1\right)\ (x<1)$ ***A1***

(or $f'(x)=e^{-x^2}\left(2x^4-4x^3-5x^2+4x+1\right)$)

$f'(x)=a\quad(x>1)$ ***A1***

substitute $x=1$ in **both** expressions and equate

$-2e^{-1}=a$ ***A1***

substitute value of a and find $b=4e^{-1}$ ***M1A1***

[8 marks]

6.

(a) f continuous $\Rightarrow \lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)$ ***M1***

$4a+2b=8$ ***A1***

$$f'(x)=\begin{cases}2, & x<2\\ 2ax+b, & 2<x<3\end{cases}$$ ***A1***

f' continuous $\Rightarrow \lim\limits_{x\to 2^-} f'(x)=\lim\limits_{x\to 2^+} f'(x)$

$4a+b=2$ ***A1***

solve simultaneously ***M1***

to obtain $a=-1$ and $b=6$ ***A1***

[6 marks]

(b) for $x\le 2$, $f'(x)=2>0$ ***A1***

for $2<x<3$, $f'(x)=-2x+6>0$ ***A1***

since $f'(x)>0$ for all values in the domain of f, f is increasing ***R1***

therefore one-to-one ***AG***

[3 marks]

(c) $x=2y-1\Rightarrow y=\dfrac{x+1}{2}$ ***M1***

$x=-y^2+6y-5\Rightarrow y^2-6y+x+5=0$ ***M1***

$y=3\pm\sqrt{4-x}$

therefore

$$f^{-1}(x)=\begin{cases}\dfrac{x+1}{2}, & x\le 3\\ 3-\sqrt{4-x}, & 3<x<4\end{cases}$$ ***A1A1A1***

Note: Award ***A1*** for the first line and ***A1A1*** for the second line.

[5 marks]

7.

(b) given continuity at $x = 5$

$5c + 2 = 16 - \frac{35}{5} \Rightarrow c = \frac{7}{5}$ ***M1A1***

[2 marks]

(c) (i) 2 ***A1***

(ii) any value ≥ 16 ***A1***

Note: Accept values less than 16 if fully justified by reference to the maximum age for a dog.

[2 marks]

(d) $\lim_{h\to 0-}\left(\frac{\frac{7}{5}(5+h)+2-\frac{7}{5}(5)-2}{h}\right)=\frac{7}{5}$ ***M1A1***

$\lim_{h\to 0+}\left(\frac{16-\frac{35}{5+h}-16+\frac{35}{5}}{h}\right)\left(=\lim_{h\to 0+}\left(\frac{\frac{-35}{5+h}+7}{h}\right)\right)$ ***M1***

$=\lim_{h\to 0+}\left(\frac{\frac{-35+35+7h}{(5+h)}}{h}\right)=\lim_{h\to 0+}\left(\frac{7}{5+h}\right)=\frac{7}{5}$ ***M1A1***

both limits equal so differentiable at $t = 5$ ***R1AG***

Note: The limits $t \to 5$ could also be used. For each value of $\frac{7}{5}$ obtained by standard differentiation award ***A1***. To gain the other 4 marks a rigorous explanation must be given on how you can get from the left and right hand derivatives to the derivative.

Note: If the candidate works with t and then substitutes $t = 5$ at the end award as follows First ***M1*** for using formula with t in the linear case, ***A1*** for $\frac{7}{5}$ Award next 2 method marks even if $t = 5$ not substituted, ***A1*** for $\frac{7}{5}$

[6 marks]