



MATH HL

OPTION - REVISION

SETS, RELATIONS AND GROUPS

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PART B: GROUPS

GROUPS

1. The binary operation  $a * b$  is defined by  $a * b = \frac{ab}{a+b}$ , where  $a, b \in \mathbb{Z}^+$ .
- (a) Prove that  $*$  is associative. (7)
  - (b) Show that this binary operation does not have an identity element. (4)
- (Total 11 marks)**

2. (a) The binary operation  $\#$  is defined on the set of real numbers by  $a \# b = a + b + 1$ .  
Show that the binary operation  $\#$  is both commutative and associative. (4)
- (b) Show that the set of real numbers forms a group under the operation  $\#$ . (4)
- (Total 8 marks)**

3. [Modified: Compilation from two past paper questions]
- Let  $T = \{\text{all real numbers except } 1\}$ . The operation  $*$  is defined on  $T$  by
- $$a * b = ab - a - b + 2, \text{ for } a, b \in T.$$
- (a) Show that  $T$  is closed under the operation  $*$ . (5)
  - (b) Show that  $*$  is associative. (5)
  - (c) Find the identity element. (3)
  - (d) Find the inverse of  $a$  under  $*$  and **hence** the inverse of 3. (3)
  - (e) In the group  $(T, *)$ 
    - (i) Prove by mathematical induction that  $\overbrace{a * a * \dots * a}^{n \text{ times}} = (a - 1)^n + 1, n \in \mathbb{Z}^+$ .  
(Note that  $\overbrace{a * a * \dots * a}^{5 \text{ times}} = a * a * a * a * a$ ).
    - (ii) Hence show that there is exactly one element in  $T$  which has finite order, apart from the identity element. Find this element and its order. (10)
- (Total 26 marks)**

4. Consider a group  $(G, \circ)$  with identity  $e$ . Suppose that  $H$  is a subset of  $G$  such that
- $$H = \{x \in G : x \circ a = a \circ x, \text{ for all } a \in G\}.$$
- Show that  $(H, \circ)$  is a subgroup of  $(G, \circ)$ , by showing that
- (a)  $e \in H$ ; (2)
  - (b) if  $x, y \in H$ , then  $x \circ y \in H$ ,  
[i.e. show that  $(x \circ y) \circ a = a \circ (x \circ y)$ ]; (5)
  - (c) if  $x \in H$ , then  $x^{-1} \in H$ . (4)
- (Total 11 marks)**

## FINITE GROUPS – CAYLEY TABLES

5.

The operation # defined on the set  $\{a, b, c, d, e\}$  has the following operation table.

#	a	b	c	d	e
a	d	c	e	a	b
b	e	d	a	b	c
c	b	e	d	c	a
d	a	b	c	d	e
e	c	a	b	e	d

Show that only three of the four group axioms are satisfied.

(Total 6 marks)

6.

The set  $S = \{a, b, c, d\}$  forms a group under each of two operations # and \*, as shown in the following group tables.

#	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

*	a	b	c	d
a	b			a
b		d		b
c				c
d	a	b		d

(a) Copy and complete the second table.

[4 marks]

(b) Solve the following equations for x.

(i)  $(b\#x)*c = d$ .

(ii)  $(a*(x\#b))*c = b$ .

[5 marks]

(Total 9 marks)

7. Consider the set  $S = \{1, 3, 4, 9, 10, 12\}$  on which the operation \* is defined as multiplication modulo 13.

(a) Write down the operation table for  $S$  under \*.

(4)

(b) Assuming multiplication modulo 13 is associative, show that  $(S, *)$  is a commutative group.

(4)

(c) State the order of each element.

(3)

(d) Find all the subgroups of  $(S, *)$

(3)

(Total 14 marks)

8.

Consider the set  $S = \{1, 3, 5, 7, 9, 11, 13, 15\}$  under the operation  $\otimes$ , multiplication modulo 16.

(a) Calculate

(i)  $3 \otimes 5$ ;

(ii)  $3 \otimes 7$ ;

(iii)  $9 \otimes 11$ .

[3 marks]

(b) (i) Copy and complete the operation table for  $S$  under  $\otimes$ .

$\otimes$	1	3	5	7	9	11	13	15
1	1	3	5	7	9	11	13	15
3	3	9				1	7	13
5	5			3	13	7	1	11
7	7		3		15	13	11	9
9	9		13	15	1		5	7
11	11	1	7	13			15	5
13	13	7	1	11	5	15	9	3
15	15	13	11	9	7	5	3	1

(ii) Assuming that  $\otimes$  is associative, show that  $(S, \otimes)$  is a group.

[5 marks]

(c) Find all elements of order

(i) 2;

(ii) 4.

[4 marks]

(d) Find a cyclic sub-group of order 4.

[2 marks]

(Total 14 marks)

9. Consider the set  $U = \{1, 3, 5, 9, 11, 13\}$  under the operation  $*$ , where  $*$  is multiplication modulo 14. (In all parts of this problem, the general properties of multiplication modulo  $n$  may be assumed.)

(a) Show that  $(3 * 9) * 13 = 3 * (9 * 13)$ .

(2)

(b) Show that  $(U, *)$  is a group.

(11)

(c) (i) Define a cyclic group.

(ii) Show that  $(U, *)$  is cyclic and find all its generators.

(9)

(d) Show that there are only two non-trivial proper subgroups of this group, and find them.

(7)

(Total 29 marks)

## PERMUTATION GROUPS

10. Let  $S$  be the group of permutations of  $\{1, 2, 3\}$  under the composition of permutations.

(a) What is the order of the group  $S$ ?

(2)

(b) Let  $p_0, p_1, p_2$ , be three elements of  $S$ , as follows:

$$p_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

List the other elements of  $S$  and show that  $S$  is not an Abelian group.

(4)

(c) Find a subgroup of  $S$  of order 3.

(2)

(Total 8 marks)

11. Let  $(S, \circ)$  be the group of all permutations of four elements  $a, b, c, d$ . The permutation that maps  $a$  onto  $c$ ,  $b$  onto  $d$ ,  $c$  onto  $a$  and  $d$  onto  $b$  is represented by

$$\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}.$$

The identity element is represented by  $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$ .

Note that  $AB$  denotes the permutation obtained when permutation  $B$  is followed by permutation  $A$ .

(a) Find the inverse of the permutation  $\begin{pmatrix} a & b & c & d \\ c & a & d & b \end{pmatrix}$ .

(1)

(b) Find a subgroup of  $S$  of order 2.

(2)

(c) Find a subgroup of  $S$  of order 4, showing that it is a subgroup of  $S$ .

(7)

(Total 10 marks)

## GROUPS AND RELATIONS (COSETS)

12. The group  $(G, \times)$  has a subgroup  $(H, \times)$ . The relation  $R$  is defined on  $G$

$$(x R y) \Leftrightarrow (x^{-1}y \in H), \text{ for } x, y \in G.$$

(a) Show that  $R$  is an equivalence relation.

(8)

(b) Given that  $G = \{e, p, p^2, q, pq, p^2q\}$ , where  $e$  is the identity element,  $p^3 = q^2 = e$ , and  $qp = p^2q$ , prove that  $qp^2 = pq$ .

(3)

(c) Given also that  $H = \{e, p^2q\}$ , find the equivalence class with respect to  $R$  which contains  $pq$ .

(5)

(Total 16 marks)

### Extra question

(d) Find all the cosets of  $H$ . (i.e. all the equivalence classes).

## HOMOMORPHISMS - ISOMORPHISMS

13. The set  $\mathbb{R}$  of all real numbers  $\mathbb{R}$  under addition is a group  $(\mathbb{R}, +)$ , and the set  $\mathbb{R}^+$  of all positive real numbers under multiplication is a group  $(\mathbb{R}^+, \times)$ . Let  $f$  denote the mapping of  $(\mathbb{R}, +)$  to  $(\mathbb{R}^+, \times)$  given by  $f(x) = 3^x$ .

(a) Show that  $f$  is an isomorphism of  $(\mathbb{R}, +)$  onto  $(\mathbb{R}^+, \times)$ .

(6)

(b) Find an expression for  $f^{-1}$ .

(1)

(Total 7 marks)

14. Let  $S = \{x \mid x = a + b\sqrt{2}; a, b \in \mathbb{Q}, a^2 - 2b^2 \neq 0\}$

(a) Prove that  $S$  is a group under multiplication,  $\times$ , of numbers.

(b) For  $x = a + b\sqrt{2}$ , define  $f(x) = a - b\sqrt{2}$ . Prove that  $f$  is an isomorphism from  $(S, \times)$  onto  $(S, \times)$ .

(Total 11 marks)

15. Let  $(\mathbb{Z}_4, +)$  denote the group whose elements are 0, 1, 2, 3, with the operation of addition of integers modulo 4. Let  $(G, *)$  denote another group of order four whose elements are  $a, b, c, d$ . Let  $\Phi$  be an isomorphism of  $(\mathbb{Z}_4, +)$  onto  $(G, *)$  defined as follows:

$$\Phi(0) = b, \Phi(1) = d, \Phi(2) = a, \Phi(3) = c.$$

(a) Write down the group table for  $(\mathbb{Z}_4, +)$ .

(1)

(b) Hence write down the group table for  $(G, *)$ .

(4)

(Total 5 marks)

16. Let  $S = \{f, g, h, j\}$  be the set of functions defined by

$$f(x) = x, g(x) = -x, h(x) = \frac{1}{x}, j(x) = -\frac{1}{x}, \text{ where } x \neq 0.$$

(a) Construct the operation table for the group  $\{S, \circ\}$ , where  $\circ$  is the composition of functions.

(3)

(b) The following are the operation tables for the groups  $\{0, 1, 2, 3\}$  under addition modulo 4, and  $\{1, 2, 3, 4\}$  under multiplication modulo 5.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

By comparing the elements in the two tables given plus the table constructed in part (a), find which groups are isomorphic. Give reasons for your answers. State clearly the corresponding elements.

(6)

(Total 9 marks)

17. (a) Let  $f_1, f_2, f_3, f_4$  be functions defined on  $\mathbb{Q} - \{0\}$ , the set of rational numbers excluding zero, such that  $f_1(z) = z$ ,  $f_2(z) = -z$ ,  $f_3(z) = \frac{1}{z}$ , and  $f_4(z) = -\frac{1}{z}$ , where  $z \in \mathbb{Q} - \{0\}$ . Let  $T = \{f_1, f_2, f_3, f_4\}$ . Define  $\circ$  as the composition of functions *i.e.*  $(f_1 \circ f_2)(z) = f_1(f_2(z))$ . Prove that  $(T, \circ)$  is an Abelian group. (6)
- (b) Let  $G = \{1, 3, 5, 7\}$  and  $(G, \diamond)$  be the multiplicative group under the binary operation  $\diamond$ , multiplication modulo 8. Prove that the two groups  $(T, \circ)$  and  $(G, \diamond)$  are isomorphic. (5)
- (Total 11 marks)**

18. Consider the following sets:

$$A = \{3^n \pmod{10} \mid n \in \mathbf{N}\}; B = \left\{ z_k = \cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2} \mid k \in \{0, 1, 2, 3\} \text{ and } i = \sqrt{-1} \right\}$$

- (a) Show that  $B$  is a group under normal multiplication. (4)
- (b) Write down the multiplication table for  $A \pmod{10}$ . (2)
- (c) Find the order of each element of  $A$ . (3)
- (d) Hence, or otherwise, show that the two groups are isomorphic. Find this isomorphism. (2)
- (Total 11 marks)**

19. ABCD is a unit square with centre O. The midpoints of the line segments [CD], [AB], [AD], [BC] are M, N, P, Q, respectively. Let  $L_1$  and  $L_2$  denote the lines (MN) and (PQ), respectively.

Consider the following symmetries of the square:

- $U$  is a clockwise rotation about O of  $2\pi$ ;
- $H$  is the reflection of the vertices of the square in the line  $L_2$ ;
- $V$  is the reflection of the vertices of the square in the line  $L_1$ ;
- $K$  is a clockwise rotation about O of  $\pi$ .

- (a) Write down the table of operations for the set  $S = \{U, H, V, K\}$  under  $\circ$ , the composition of these geometric transformations. (4)
- (b) Assuming that  $\circ$  is associative, prove that  $(S, \circ)$  forms a group. (4)

Consider the set  $C = \{1, -1, i, -i\}$  and the binary operation  $\diamond$  defined on  $C$ , where  $\diamond$  is the multiplication of complex numbers.

- (c) Find the operation table for the group  $(C, \diamond)$ . (3)
- (d) Determine whether the groups  $(S, \circ)$  and  $(C, \diamond)$  are isomorphic. Give reasons for your answer. (4)
- (Total 15 marks)**

## THEORETICAL

20. Let  $(H, \times)$  be a group and let  $a$  be one of its elements such that  $a \times a = a$ . Show that  $a$  must be the identity element of the group. (Total 4 marks)
21. (a) State Lagrange's theorem.  
 (b) Let  $(G, \circ)$  be a group of order 24 with identity element  $e$ . Let  $a \in G$ , and suppose that  $a^{12} \neq e$  and  $a^8 \neq e$ . Prove that  $(G, \circ)$  is a cyclic group with generator  $a$ . (Total 7 marks)
22. (a) Explain what is meant by a cyclic group. (2)  
 Let  $(G, \#)$  be a finite group such that its order  $p$  is a prime number.  
 (b) Show that  $(G, \#)$  is cyclic. (5)  
(Total 7 marks)
23. Let  $(G, *)$  be a group, and  $H$  a subset of  $G$ . Given that for all  $a, b \in H$ ,  $a^{-1} * b \in H$ , prove that  $(H, *)$  is a subgroup of  $(G, *)$ . (Total 6 marks)
24. The group  $(G, *)$  is defined on the set  $\{e, a, b, c\}$ , where  $e$  denotes the identity element. Prove that  $a * b = b * a$ . (Total 6 marks)
25. Let  $(G, *)$  be a group where  $*$  is a binary operation on  $G$ . The identity element in  $G$  is  $e$ , such that  $G \neq \{e\}$ . The group  $G$  is cyclic, and its only subgroups are  $\{e\}$  and  $G$ . Prove that  $G$  is a finite cyclic group of prime order. (Total 6 marks)
26. Let  $(G, \circ)$  be a group with subgroups  $(H, \circ)$  and  $(K, \circ)$ . Prove that  $(H \cup K, \circ)$  is a subgroup of  $(G, \circ)$  if and only if one of the sets  $H$  and  $K$  is contained in the other. (Total 8 marks)
27. (a) Define an isomorphism between two groups  $(G, \circ)$  and  $(H, \bullet)$ . (2)  
 (b) Let  $e$  and  $e'$  be the identity elements of groups  $G$  and  $H$  respectively. Let  $f$  be an isomorphism between these two groups. Prove that  $f(e) = e'$ . (4)  
 (c) Prove that an isomorphism maps a finite cyclic group onto another finite cyclic group. (4)  
(Total 10 marks)
28. Let  $a, b$  and  $p$  be elements of a group  $(H, *)$  with an identity element  $e$ .  
 (a) If element  $a$  has order  $n$  and element  $a^{-1}$  has order  $m$ , then prove that  $m = n$ . (5)  
 (b) If  $b = p^{-1} * a * p$ , prove, by mathematical induction, that  $b^m = p^{-1} * a^m * p$ , where  $m = 1, 2, \dots$  (4)  
(Total 9 marks)
29. (a) In any group, show that if the elements  $x, y$ , and  $xy$  have order 2, then  $xy = yx$ . (5)  
 (b) Show that the inverse of each element in a group is unique. (3)  
 (c) Let  $G$  be a group. Show that the correspondence  $x \leftrightarrow x^{-1}$  is an isomorphism from  $G$  onto  $G$  if and only if  $G$  is **abelian**. (4)  
(Total 12 marks)

**THEORY – PROOFS**

30. Let  $(G, \circ)$  be a group with subgroups  $H$  and  $K$ .  
 (a) Prove that  $H \cap K$  is a subgroup of  $G$ .  
 (b) Show by a counterexample that  $H \cup K$  is not necessarily a subgroup.  
 [hint: Use  $(\mathbb{Z}, +)$  as a group. The set of multiples of any fixed number is a subgroup]

31. Prove the following  
 (a) for a binary operation, if an identity element exists it is unique.  
 (b) given that the operation is associative, if  $x$  has an inverse this is unique.

32. Left and right cancelation: in a group  $G$   
 (a)  $a * x = a * y \Rightarrow x = y$   
 (b)  $x * a = y * a \Rightarrow x = y$

33. Any cyclic group is Abelian.

34. Suppose that  $(G, *)$  is a group and  $H$  is a non-empty subset of  $G$ . Then  
 (a)  $H$  is a subgroup of  $G$  if  $a, b \in H \Rightarrow a * b^{-1} \in H$   
 (b) provided that  $G$  is finite,  $H$  is a subgroup of  $G$  if  $H$  is closed under  $*$ .

35. In a finite group, the order of any element divides the order of the group .

36. Let  $(G, *)$  and  $(H, \bullet)$  be two groups with identities  $e_G$  and  $e_H$  respectively. If

$$f: G \rightarrow H \text{ is a homomorphism}$$

then

- (a)  $f(e_G) = e_H$  [f maps identity to identity]  
 (b)  $f(a^{-1}) = f(a)^{-1}$  [f maps inverse to inverse]

Given that  $f$  is an isomorphism

- (c)  $f$  maps an element of order  $n$  to an element of order  $n$ .  
 (d) If  $G$  is cyclic then  $H$  is cyclic.  
 (e) IF  $G$  is abelian then  $H$  is abelian

37. Let  $(G, *)$  and  $(H, \bullet)$  be two groups with identities  $e_G$  and  $e_H$  respectively. If

$$f: G \rightarrow H \text{ is a homomorphism}$$

then

- (a) The kernel  $\ker f$  is a subgroup of  $G$ .  
 (b) The range  $f(G)$  is a subgroup of  $H$ .  
 (c)  $f$  is one-to-one if and only if  $\ker f = \{ e_G \}$  [consists only of  $e_G$ ]  
 (Thus,  $f$  is an isomorphism if and only if  $\ker f = \{ e_G \}$  and  $f(G) = H$ )

38. Give the definitions of the following terms:

Binary operation	Group	Order of a group: $ G $	Homomorphism
Closed subset	Subgroup	Order of an element: $\text{order}(a)$	Isomorphism
Identity element	Latin square	Cyclic group - generator	Kernel of $f$
Inverse of $x$	Cayley table	Cosets: $aH$ or $Ha$	Range of $f$

Give examples.