MATH HL
OPTION
REVISION - SOLUTIONS
SETS, RELATIONS AND GROUPS
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PART A: SETS AND RELATIONS

SETS

1. Venn diagrams are

\[ A - B \]
\[ A \cap (A - B) \]
\[ B - A \]
\[ B \cap (B - A) \]

Note: Award (A1) if both the Venn diagrams are correct otherwise award (A0).

From the Venn diagrams, we see that \( B \cap (A - B) = \phi \) and \( B \cap (B - A) = B - A \) (M1)
Hence they are not equal. (C1)

Note: Award (M0)(C1) if no reason is given. Accept other correct diagrams.

2. (a) \((A \cup B)'\) is given by

\[ A' \cup B' \]

\( A' \cap B' \) is given by

\[ A' \cap B' \]

Hence \((A \cup B)' = A' \cap B'\). (AG) 2

(b) \[ (A' \cup B) \cap (A \cup B') \]
\[ = (A' \cup B') \cup (A \cup B')' \] (A1)
\[ = (A \cap B') \cup (A' \cap B) \] (A1)
\[ = [(A \cap B') \cup A'] [(A \cap B') \cup B] \] (M1)
\[ = [(A \cap A') \cap (B' \cup A') \cap [(A \cup B) \cap (B' \cup B)] \]
\[ = (B' \cup A') \cap (A \cup B) \] (A1)
\[ = (A \cup B)' \cap (A \cup B) \] (AG) 2
3. (a) The shaded area denotes \(A - B\) and \(A \cap B'\) confirming that \(A - B = A \cap B'\) (AG) 1

(b) \[
A - (B \cup C) = A \cap (B \cup C)' \\
= A \cap (B' \cap C) \\
= A \cap B' \cap C \\
\]
\((A - B) \cap (A - C) = (A \cap B') \cap (A \cap C') \\
= A \cap B' \cap A \cap C' \\
= A \cap A \cap B' \cap C' \\
= A \cap B' \cap C' \\
\] (A1) 4

4. (a)

(b) \[
(A \cup B) - (B \cap A) = (A \cup B) \cap (B \cap A)' \\
= [A \cap (B \cap A)'] \cup [B \cap (B \cap A)'] \\
= [A \cap (B' \cup A')'] \cup [B \cap (B' \cup A')'] \\
= (A \cap B') \cup (A \cap A') \cup (B \cap B') \cup (B \cap A') = (A \cap B') \cup (B \cap A') \\
= (A - B) \cup (B - A) \\
\] (A1) (A1) 2

5. (a)

Hence \(A \cup (B \cap A')' = A \cup B'\) (AG)

(b) \[
\left((A \cap B)' \cup B\right)' = \emptyset \\
\Rightarrow \left((A \cap B)' \cup B\right)' = \emptyset \\
\] (AI) (RI) (AG) [4]
6. \[ A \Delta B = (A \cap B) \cup (B \cap A) \]
   \[ = (A \cap B') \cup (B \cap A') \quad \text{M1A1} \]
   \[ = ((A \cap B') \cup B) \cap ((A \cap B') \cup A') \quad \text{M1A1} \]
   \[ = ((A \cup B) \cap (B' \cup B)) \cap ((A \cup A') \cap (B' \cup A')) \quad \text{M1A1} \]
   \[ = (A \cup B) \cap (A' \cup B') \quad \text{A1} \]
   \[ = (A \cup B) \cap (A \cap B')' \quad \text{A1} \]

   \textbf{Note: Illustration using a Venn diagram is not a proof.} \[6\]

7. (a)

   \[ A \cap B = (A \cap B) \cap (A \cup B) \]
   \[ = (A \cap B) \cap (A \cap B') \quad \text{M1A1} \]
   \[ = A \quad \text{A1} \]

   (b) From part (a) \( (A' \cap B) \cup C' = (A' \cup C') \cap (B' \cup C') \).

   From De Morgan’s laws \( (A \cap C)' = A' \cup C \) and \( (B \cap C)' = B \cup C' \).

   So \( (A' \cap B) \cup C' = (A \cap C)' \cap (B' \cap C)' \).

   \text{That is,} \( (A \cap B) \cup C = (A \cap C) \cap (B \cup C) \). \[5\]

8. By definition of \( \bullet \) and de Morgan’s laws,

   \( (X \bullet Y)' = (X \cap Y)' \cap (X' \cap Y') \)
   \[ = (X' \cap Y') \cap (X \cup Y) \quad \text{M1} \]
   \[ = (X \cup Y) \cap (X' \cup Y') \quad \text{R1} \]

9. (a) \( A \# A = A' \cup A' = A' \) \[3\]

(b) \( (A \# A) \# (B \# B) = A' \# B' = (A')' \cap (B')' = A \cup B \) \[1\]

(c) \( (A \# B) \# (A \# B) = (A' \cup B') \# (A' \cup B') \)
   \[ = (A' \cup B')' \quad \text{A1} \]
   \[ = A \cap B \text{ (by de Morgan’s law)} \quad \text{AG} \]

\[6\]

**RELATIONS**

10. (a) Since the main diagonal of the matrix has ones, this means that
    every element is related to itself and consequently the relation is
    reflexive. \[C1\]

    Also, the matrix is symmetric and hence, the relation is symmetric. \[C2\]

(b) The partition of A is the set of all equivalent classes.

    The three classes are \( \{\{a, c, e\}, \{b, d\}, \{f\}\} \) \[A3\]

[7]
11. (a) \( \gcd(a, a) = a > 1 \), since \( a \in S \).

  Hence \( R \) is reflexive. (A1) (AG) 1

(b) Since \( \gcd(a, b) = \gcd(b, a) \),

\( \gcd(a, b) > 1 \Rightarrow \gcd(b, a) > 1 \)

  Hence \( R \) is symmetric (AG) 2

(c) Any correct counter example e.g.

\( \gcd(25, 15) = 5 \Rightarrow 25 \not R 15 \)
\( \gcd(15, 21) = 3 \Rightarrow 15 \not R 21 \)
\( \gcd(25, 21) = 1 \Rightarrow 25 \not R 21 \)

  Hence \( R \) is not transitive (AG) 3

12. (a) \( R \) is reflexive because \( |z| = |z| \Rightarrow zRz \).

  (A1)

\( R \) is symmetric because \( |z_1| = |z_2| \Rightarrow |z_2| = |z_1| \Rightarrow (z_1, Rz_2 \Rightarrow z_2, Rz_1) \)

  ⇒ (A1)

\( R \) is transitive because \( |z_1| = |z_2| \) and \( |z_2| = |z_3| \Rightarrow |z_1| = |z_3| \)

  ⇒ (A1) 3

(b) In the Argand diagram this corresponds to the concentric circles centered at the origin.

  (A1) (AG) 2

13. (a) To show that the relation is an equivalence relation we have to show that it is:

  Reflexive: \( (a, b) \Delta (a, b) \) since \( a^2 + b^2 = a^2 + b^2 \)

  (R1)

  Symmetric: \( (a, b) \Delta (c, d) \Rightarrow a^2 + b^2 = c^2 + d^2 \Leftrightarrow c^2 + d^2 = a^2 + b^2 \Leftrightarrow (c, d) \Delta (a, b) \)

  (R1)

  Transitive: \( (a, b) \Delta (c, d) \) and \( (c, e) \Delta (e, f) \Leftrightarrow a^2 + b^2 = c^2 + d^2 \) and \( c^2 + d^2 = e^2 + f^2 \Leftrightarrow a^2 + b^2 = e^2 + f^2 \Leftrightarrow (a, b) \Delta (e, f) \)

  (R2)

(b) This is the set of ordered pairs \( (x, y) \) such that \( x^2 + y^2 = 5 \).

  (R1)(A1)

  Notes: It is a circle with radius \( \sqrt{5} \).

(c) The partition is the set of all concentric circles in the plane with the origin as the centre.

  (R1)(A1) [7]
15. (a) To show that $R$ is an equivalence relation, we show it is reflexive, symmetric, transitive.

Reflexivity: Since $ab = ba$ for $a, b \in \mathbb{Z}$, we have $(a, b) R (a, b)$.

(Symmetry: $(a, b) R (c, d) \iff ad = bc \iff da = cb \iff cb = da$

$(c, d) R (a, b)$

Transitivity: $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow ad = bc$ and $cf = ed$.

If $c = 0$, $ad = 0$ and $ed = 0$. Since $d \neq 0$, $a = 0$ and $e = 0$.

$\Rightarrow af = be \Rightarrow (a, b) R (e, f)$.

If $c \neq 0$, $adcf = bced$ i.e. $(af)dc = (be)cd$ or $(af)cd = (be)cd$

i.e. $af = be \Rightarrow (a, b) R (e, f)$, since $cd \neq 0$

Note: Award (M0)(R1) if $cd \neq 0$ is not mentioned.

(b) $ad = bc \iff a : b = c : d$

i.e. the classes are those pairs $(a, b)$ and $(c, d)$ with $\frac{a}{b} = \frac{c}{d}$

i.e. the elements of those pairs are in the same ratio.

i.e. the elements are on the same line going through the origin.

16. (a) $(a, b) R (p, q) \Rightarrow \max (|a|, |b|) = \max (|p|, |q|)$

$\Rightarrow R$ is symmetric

$(a, b) R (a, b) \Rightarrow \max (|a|, |b|) = \max (|a|, |b|)$

$R$ is reflexive

$(a, b) R (x, y)$ and $(x, y) R (p, q) \Rightarrow (a, b) R (p, q)$

since $\max (|a|, |b|) = \max (|x|, |y|)$ and $\max (|x|, |y|) = \max (|p|, |q|)$

$\Rightarrow \max (|a|, |b|) = \max (|p|, |q|)$

$R$ is transitive.

$\Rightarrow R$ is an equivalence relation.

(b) (i) If $\max (|x|, |y|) = c$

Then $|x| = c$ and $|y| \leq c$

$\Rightarrow x = \pm c$ and $-c \leq y \leq c$

$\text{or } |y| = c$ and $|x| \leq c$

$\Rightarrow y = \pm c$ and $-c \leq x \leq c$

(ii) i.e. Concentric squares with a centre at $(0, 0)$

17. (a) $\forall a \in \mathbb{Z}, a R a$

$\forall a, b \in \mathbb{Z}, a R b \Rightarrow m$ divides $a - b$ $\Rightarrow m$ divides $b - a$ $\Rightarrow b R a$

$\forall a, b, c \in \mathbb{Z}, a R b$ and $b R c \Rightarrow m$ divides $(a - b)$ and $m$ divides $(b - c)$

$m$ divides $(a - b) + (b - c) \Rightarrow m$ divides $(a - c) \Rightarrow a R c$

Hence $R$ is an equivalence relation.

(b) For any reasonable attempt to explain that the equivalence relation partitions the set.

For either the list of equivalence classes that partition $\mathbb{Z}$ or an attempt to explain that there are $m$ equivalence classes.
18. (a) \(aRa\) since \(a^2 - a^2 = 0 \equiv 0 \pmod{5}\) \hspace{1cm} (A1)  \\
aRb \Rightarrow bRa\) since \(a^2 - b^2 = 0 \pmod{5}\) \Rightarrow b^2 - a^2 \equiv 0 \pmod{5}\) \hspace{1cm} (A1)  \\
adRb\) and \(bRc \Rightarrow aRc\) since \(a^2 - b^2 \equiv 0 \pmod{5}\) and \(b^2 - c^2 \equiv 0 \pmod{5}\)  \\
\Rightarrow a^2 - c^2 = a^2 - b^2 + b^2 - c^2 \equiv 0 \pmod{5}\) \hspace{1cm} (A2)  \\
Hence \(R\) is an equivalence relation. \hspace{1cm} (AG)  

(b) (i) It is the set of all the elements \(b\) of \(Y\) such that \(bRa\).  \\
\hspace{1cm} (or equivalent) \hspace{1cm} (C2)  \\
(ii) \(\{5,10\}\) \hspace{1cm} (A1)  \\
\hspace{1cm} \{1,4,6,9\}\hspace{1cm} (A1)  \\
\hspace{1cm} \{2,3,7,8\} \hspace{1cm} (A1)  

19. (a) Reflexive: \(7^a \equiv 7^a\) (modulo 10) so \(aRa\) \hspace{1cm} (A1)  \\
Symmetric: \(7^a \equiv 7^b\) (modulo 10) \Rightarrow \(7^b \equiv 7^a\) (modulo 10) so \(aRb \Rightarrow bRa\) \hspace{1cm} (A1)  \\
Transitive: Let \(7^a \equiv 7^b\) (modulo 10) and \(7^b \equiv 7^c\) (modulo 10) \hspace{1cm} (M1)  \\
Then, \(7^a = 7^b + 10\lambda\) and \(7^b = 7^c + 10\mu\)  \\
so \(7^a = 7^c + 10(\lambda + \mu)\) so \(aRb\) and \(bRc \Rightarrow aRc\) \hspace{1cm} (A1)  

(b) We note that \(7^0 = 1, 7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401\)  \\
The equivalence classes are therefore \hspace{1cm} (A1)  \\
\hspace{1cm} 0, 4, 8, …  \\
\hspace{1cm} 1, 5, 9, …  \\
\hspace{1cm} 2, 6, 10, …  \\
\hspace{1cm} 3, 7, 11, …  \\
\hspace{1cm} 4  

(c) \(7^{503}\) (modulo 10) \(\equiv 7^3\) (modulo 10) = 3. \hspace{1cm} (A1)  

20. We show that \(S\) is a reflexive, symmetric and transitive relation on \(X\).  
Since \(R\) is an equivalence relation on \(Y\), it is reflexive, symmetric, and transitive. \hspace{1cm} (R2)  

For all \(a\) in \(X\), reflexivity of \(R\) implies \(h(a) Rh(a)\). By the definition of the relation \(S\) on \(X\), \(a S a\) for all \(a\) in \(X\). Hence, \(S\) is reflexive. \hspace{1cm} (R2)  

Let a \(S \ b\) \hspace{1cm} (R2)  

Let a \(S \ b\) \hspace{1cm} (R2)  

Let a \(S \ b\) and \(b S c\) for any a, b, c in \(X\). Then h(a) \(R\) h(b) and \(h (b) \ R \ h \ (c)\).  
Since R is a transitive relation, we get h(a) \(R\) h(c) \hspace{1cm} (M1)  
By definition of the relation \(S\) on \(X\), \(a S c\). Thus \(S\) is transitive on \(X\). \hspace{1cm} (R1)  

21. (a) reflexive: \(f R f\), since \(f = I f I^{-1}\), where \(I\) is the identity function. \hspace{1cm} (6)  

\textbf{symmetric:} \hspace{1cm} g = h^{-1} of h \hspace{1cm} (where h is a bijective function) \hspace{1cm} (where h is a bijective function) \hspace{1cm} (since \(h^{-1}\) is also bijective function)  

\textbf{transitive:} \hspace{1cm} f R g and gR k \Rightarrow f = h_1 of h_2^{-1} \hspace{1cm} (since \(h_1 \circ h_2^{-1}\) is also bijective function) \hspace{1cm} (since \(h_1 \circ h_2^{-1}\) is also bijective function) \hspace{1cm} (since \(h_1 \circ h_2^{-1}\) is also bijective function)  

\textbf{We find the related function:} \hspace{1cm} (h o h^{-1})(x) = 2(x-1)+1 = 2x-1  

(b) \(f(x) = 2x\) \hspace{1cm} (12)  

If we consider the bijective function \(h(x) = x+1\), then \(h^{-1}(x) = x-1\)
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22. \( f(n) = f(n') \), for any \( n, n' \) in \( \mathbb{N} \), implies \( n + 1 = n' + 1 \).
   Hence \( n = n' \). Hence \( f \) is an injection from \( \mathbb{N} \) to \( \mathbb{N} \).
   \( \text{(R1)} \)
   There is no point in the domain of \( f \) which is mapped to zero.
   Hence \( f \) is not a surjection.
   \( \text{(R1)} \)

23. A bijection is both one-to-one and onto, so by considering a sketch of each function
   
   \[
   \begin{align*}
   y &= x^2 + 1 \\
   y &= x^3 \\
   y &= x
   \end{align*}
   \]
   
   we can see that for \( \mathbb{R} \) to \( \mathbb{R} \) only \( y = x^3 \) is one-to-one and onto.
   \( \text{(A1)} \)

24. (a) If the function is injective, then \( f(x, y) = f(a, b) \) must imply that \( (x, y) = (a, b) \).
   \( \text{(R1)} \)
   \[
   f(x, y) = f(a, b) \iff (2y - x, x + y) = (2b - a, a + b)
   \]
   \( \iff 2y - x = 2b - a \text{ and } x + y = a + b \iff 3y = 3b \iff y = b, x = a \)
   \( \text{(A1)} \)
   \( \iff (x, y) = (a, b) \)
   \( \text{(R1)} \)

(b) If the function is surjective, then given \( (u, v) \in \mathbb{R}^2 \), we should be able to find \( (x, y) \in \mathbb{R}^2 \) such that \( f(x, y) = (u, v) \).
   \( \text{(R1)} \)
   \[
   f(x, y) = (u, v) \iff (2y - x, x + y) = (u, v)
   \]
   \( \iff 2y - x = u \text{ and } x + y = v \iff y = \frac{u + v}{3}, x = \frac{2v - u}{3} \)
   \( \text{(R1)} \)

(c) Since \( f \) is injective and surjective, it is bijective. Since every bijective function has an inverse, then \( f \) has an inverse.
   \( \text{(R1)} \)

From the last line of the previous part, replace \( u \) by \( x \) and \( v \) by \( y \):

\[
 f^{-1}(x, y) = \left( \frac{2y - x}{3}, \frac{x + y}{3} \right)
\]
   \( \text{(A1)} \)

Now \( f^{-1}(f(x, y)) = \left( \frac{2(2y - x) - (2y - x)}{3}, \frac{2(2y - x) + (x + y)}{3} \right) \)
   \( = \left( \frac{3x}{3}, \frac{3y}{3} \right) = (x, y) \)
   \( \text{(R1)} \)

25. (a) (i) \( f \) is an increasing function so it is injective.
   \( \text{R1 A1} \)

(ii) Let \( f(n) = 1 \) (or any other appropriate value)
   \( \text{M1} \)
   Then \( 5n + 4 = 1, n = \frac{3}{5} \) which is not in the domain
   \( \therefore f \) is not surjective.
   \( \text{A1} \)

(b) \( g(x, y) = (x + 2y, 3x - 5y) \)
   \( \text{M1} \)

(i) Let \( g(x, y) = g(s, t) \) so \( (x + 2y, 3x - 5y) = (s + 2t, 3s - 5t) \)
   \( \text{M1} \)
   \[
   x + 2y = s + 2t, 3x - 5y = 3s - 5t
   \]
   \( y = t \text{ and } x = s \Rightarrow (x, y) = (s, t) \)
   \( g \) is injective.
   \( \text{A1} \)
(ii) Let \((u, v)\) be an element of the codomain.
\[ x + 2y = u, \quad 3x - 5y = v \]
Then \(-11y = -3u + v\) so \(y = \frac{3u - v}{11}\) \hspace{1cm} (A1)
and \(11x = 5u + 2v\) so \(x = \frac{5u - 2v}{11}\) \hspace{1cm} (A1)
Since \(\left( \frac{5u + 2v}{11}, \frac{3u - v}{11} \right)\) is in the domain then \(g\) is surjective \hspace{1cm} (R1). 

(c) \(g^{-1}(x, y) = \left( \frac{5x + 2y}{11}, \frac{3x - y}{11} \right)\) \hspace{1cm} (A2)

\[ 26. \]

(a) We need to show that \(f\) is surjective and injective. \hspace{1cm} (R1)
It is surjective, all elements of \(S\) are images. \hspace{1cm} (R1)
It is injective, 1:1 function. \hspace{1cm} (R1)
So \(f\) is a bijection. \hspace{1cm} (AG)

\[ 27. \]

There is \(\binom{n}{0}\) empty subset. \hspace{1cm} (A1)

There are \(\binom{n}{1}\) subsets with 1 element. \hspace{1cm} (A1)

There are \(\binom{n}{2}\) subsets with 2 elements. 

\[ \ldots \]

There are \(\binom{n}{k}\) subsets with \(k\) elements. \hspace{1cm} (A1)

So in total there are \(\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}\) \hspace{1cm} (M1)(A1)
\[ = (1 + 1)^n = 2^n \] subsets. \hspace{1cm} (A1)(AG)

\[ \text{OR} \]

Since each of the \(n\) elements in set \(X\) can be either included in the subset or not, there are \(2^n\) possible subsets. \hspace{1cm} (A6)

\[ 28. 29. 30. \] Answers can be found in the lecture notes.