MATH HL

OPTION

REVISION - SOLUTIONS

SETS, RELATIONS AND GROUPS

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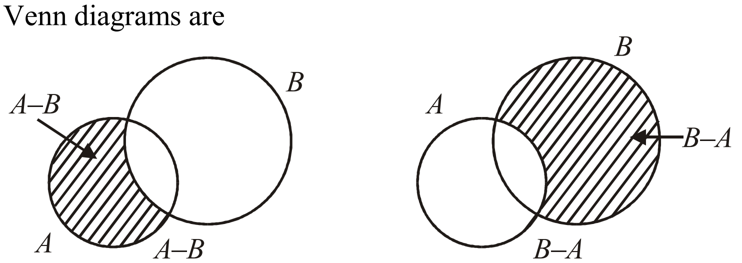
PART A: SETS AND RELATIONS

SETS

1. Venn diagrams are



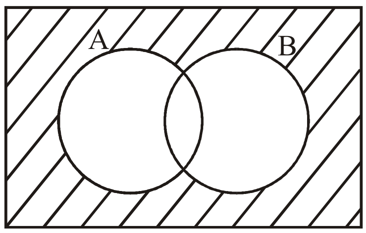
(A1)

Note: *Award (A1) if both the Venn diagrams are correct otherwise award (A0).*

From the Venn diagrams, we see that $B\cap(A-B)=\phi$ and $B\cap(B-A)=B-A$ (M1)
Hence they are not equal. (C1)
Note: *Award (M0)(C1) if no reason is given. Accept other correct diagrams.*

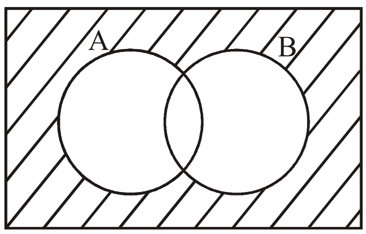
[3]

2. (a) $(A\cup B)'$ is given by



(A1)

$A'\cap B'$ is given by



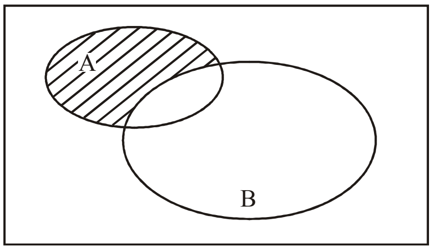
(A1)

Hence $(A\cup B)'=A'\cap B'$. (AG) 2

(b) $[(A'\cup B)\cap(A\cup B')]'=(A'\cup B)'\cup(A\cup B')'$ (A1)
$=(A\cap B')\cup(A'\cap B)$ (A1)
$=[(A\cap B')\cup A']\ [(A\cap B')\cup B)]$ (M1)
$=[(A\cup A')\cap(B'\cup A']\cap[(A\cup B)\cap(B'\cup B)]$
$=(B'\cup A')\cap(A\cup B)$ (A1)
$=(A\cap B)'\cap(A\cup B)$ (AG) 2

[6]

3. (a)



The shaded area denotes $A - B$ and $A \cap B'$ (A1)

confirming that $A - B = A \cap B'$ (AG) 1

(b) $A - (B \cup C) = A \cap (B \cup C)'$ (M1)

$= A \cap (B' \cap C')$

$= A \cap B' \cap C'$ (A1)

$(A - B) \cap (A - C) = (A \cap B') \cap (A \cap C')$ (M1)

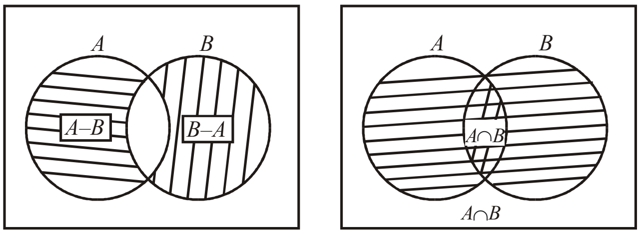
$= A \cap B' \cap A \cap C'$

$= A \cap A \cap B' \cap C'$

$= A \cap B' \cap C'$ (A1) 4

[5]

4. (a)



(A1)(A1) 2

(b) $(A \cup B) - (B \cap A) = (A \cup B) \cap (B \cap A)'$

$= [A \cap (B \cap A)'] \cup [B \cap (B \cap A)']$ (A1)

$= [A \cap (B' \cup A')] \cup [B \cap (B' \cup A')]$ (M1)

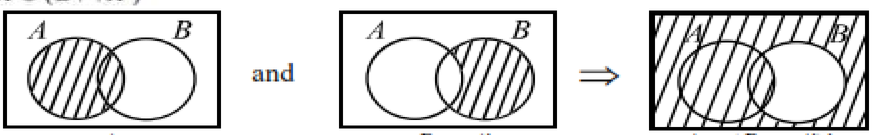
$= (A \cap B') \cup (A \cap A') \cup (B \cap B') \cup (B \cap A') = (A \cap B') \cup (B \cap A')$ (M1)

$= (A - B) \cup (B - A)$ (A1) 4

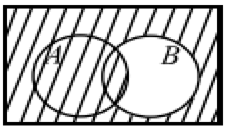
[6]

5.

(a) $A \cup (B \cap A')'$



A $\quad$ $B \cap A'$ $\quad$ $A \cup (B \cap A')'$

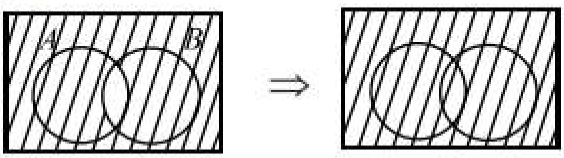


$A \cup B'$

(M1)(A1)

Hence $A \cup (B \cap A')' = A \cup B'$ ***(AG)***

(b) $((A \cap B)' \cup B)' = \varnothing$



(A1)

$(A \cap B)'$ $\quad$ $(A \cap B)' \cup B$

everything shaded ***(R1)***

$\Rightarrow ((A \cap B)' \cup B)' = \varnothing$ ***(AG)***

[4]

6. $A\Delta B = (A \backslash B) \cup (B \backslash A)$

$= (A \cap B') \cup (B \cap A')$

$= ((A \cap B') \cup B) \cap ((A \cap B') \cup A')$ M1A1

$= ((A \cup B) \cap (B' \cup B) \cap ((A \cup A') \cap (B' \cup A'))$ M1A1

$= ((A \cup B) \cap U) \cap (U \cap (B' \cup A')$ A1

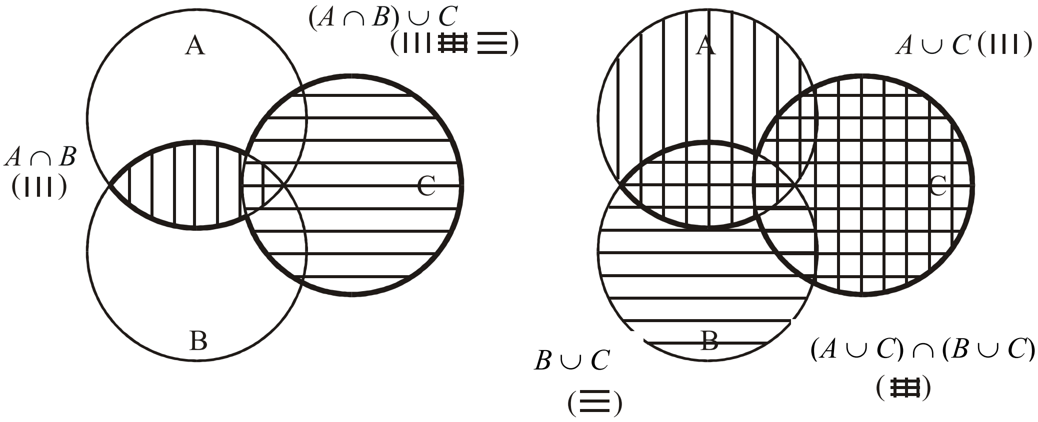
$= (A \cup B) \cap (A' \cup B')$

$= (A \cup B) \cap (A \cap B)'$ A1

Note: *Illustration using a Venn diagram is not a proof.*

[6]

7. (a)



That is, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ (M1)(A1) 2

(b) From part (a) $(A' \cap B) \cup C' = (A' \cup C') \cap (B \cup C')$. (A1)
From De Morgan's laws $(A \cap C)' = A' \cup C'$, and $(B' \cap C)' = B \cup C'$ (A1)(A1)
So $(A' \cap B) \cup C' = (A \cap C)' \cap (B' \cap C)'$ (AG) 3

[5]

8. By definition of • and de Morgan's laws,
$(X \bullet Y)' = (X \cap Y)' \cap (X' \cap Y')'$ (M1)
$= (X' \cup Y') \cap (X \cup Y)$ (M1)
$= (X \cup Y) \cap (X' \cup Y')$ (R1) 3

[3]

9. (a) $A \# A = A' \cup A' = A'$ (A1)(AG) 1

(b) $(A \# A)\#(B \# B) = A' \# B' = (A')' \cup (B')' = A \cup B$ (M1)(A1)(AG) 2

(c) $(A \# B)\#(A \# B) = (A' \cup B')\#(A' \cup B')$ (M1)(A1)
$= (A' \cup B')'$ (A1)
$= A \cap B$ (by de Morgan's law) (AG) 3

[6]

RELATIONS

10. (a) Since the main diagonal of the matrix has ones, this means that every element is related to itself and consequently the relation is reflexive. (C1)
Also, the matrix is symmetric and hence, the relation is symmetric. (C2) 3

(b) The partition of A is the set of all equivalent classes. (C1)
The three classes are $\{\{a, c, e\}, \{b, d\}, \{f\}\}$ (A3) 4

[7]

11. (a) $\gcd(a, a) = a > 1$, since $a \in S$. (A1)
Hence R is reflexive. (AG) 1

(b) Since $\gcd(a, b) = \gcd(b, a)$, (M1)
$\gcd(a, b) > 1 \Rightarrow \gcd(b, a) > 1$ (A1)
Hence R is symmetric (AG) 2

(c) Any correct counter example e.g.
$\gcd(25, 15) = 5 \Rightarrow 25 R\, 15$ (A1)
$\gcd(15, 21) = 3 \Rightarrow 15 R\, 21$ (A1)
$\gcd(25, 21) = 1 \Rightarrow 25$ not $R\, 21$ (A1)
Hence R is not transitive (AG) 3

[6]

12. (a) R is reflexive because $|z| = |z| \Rightarrow zRz$. (A1)
R is symmetric because $(|z_1| = |z_2| \Rightarrow |z_2| = |z_1|) \Rightarrow (z_1 R z_2 \Rightarrow z_2 R z_1)$ (A1)
R is transitive because $(|z_1| = |z_2| \text{ and } |z_2| = |z_3| \Rightarrow |z_1| = |z_3|)$
$\Rightarrow (z_1 R z_2 \text{ and } z_2 R z_3 \Rightarrow z_1 R z_3)$ (A1) 3

(b) In the Argand diagram this corresponds to the concentric circles (A1)
centered at the origin. (A1) 2

[5]

13. (a) To show that the relation is an equivalence relation we have to show that it is:

Reflexive: $(a, b)\ \Delta\ (a, b)$ since $a^2 + b^2 = a^2 + b^2$ (R1)

Symmetric: $(a, b)\ \Delta\ (c, d) \Rightarrow a^2 + b^2 = c^2 + d^2 \Leftrightarrow$
$c^2 + d^2 = a^2 + b^2 \Leftrightarrow (c, d)\Delta(a, b)$ (R1)

Transitive: $(a, b)\Delta(c, d)$ and $(c, d)\Delta(e, f) \Leftrightarrow$
$a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2 \Leftrightarrow$
$a^2 + b^2 = e^2 + f^2 \Leftrightarrow (a, b)\Delta(e, f)$ (R2)

(b) This is the set of ordered pairs (x, y) such that $x^2 + y^2 = 5$. (R1)(A1)
***Notes:** It is a circle with radius* $\sqrt{5}$.

(c) The partition is the set of all concentric circles in the plane with the origin as the centre. (R1)(A1)

[7]

14.

(a) Reflexivity: $(x_1, y_1)R(x_1, y_1)$ since $x_1 + y_1 = x_1 + y_1$ *(A1)*
Symmetry: $(x_1, y_1)R(x_2, y_2) \Rightarrow (x_2, y_2)R(x_1, y_1)$ since $x_1 + y_2 = x_2 + y_1$
$\Rightarrow x_2 + y_1 = x_1 + y_2$ *(A1)*
Transitivity: Suppose that $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$. Then,
$x_1 + y_2 = x_2 + y_1$ and $x_3 + y_2 = x_2 + y_3$ *(M1)*
Subtracting, $x_1 - x_3 = y_1 - y_3$ or $x_1 + y_3 = x_3 + y_1$ *(A1)*
It follows that $(x_1, y_1)R(x_3, y_3)$.

[4 marks]

(b) $x_1 + y_2 = x_2 + y_1 \Rightarrow y_1 - x_1 = y_2 - x_2$ *(M1)*
The equivalence classes are lines with equations $y = x + \text{Constant}$. *(A1)*

[2 marks]

[6]

15. (a) To show that R is an equivalence relation, we show it is reflexive, symmetric, transitive.

Reflexivity: Since $ab = ba$ for $a, b \in \mathbb{Z}$, we have $(a, b)\ R\ (a, b)$. (A1)

Symmetry: $(a, b)\ (c, d) \Leftrightarrow ad = bc \Leftrightarrow da = cb \Leftrightarrow cb = da$
$(c, d)\ R\ (a, b)$ (A1)

Transitivity: $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow ad = bc$ and $cf = ed$.
If $c = 0$, $ad = 0$ and $ed = 0$. Since $d \neq 0$, $a = 0$ and $e = 0$. (M1)
$\Rightarrow af = be \Rightarrow (a, b)\ R\ (e, f)$.
If $c \neq 0$, $adcf = bced$ i.e. $(af)dc = (be)cd$ or $(af)cd = (be)cd$
i.e. $af = be \Rightarrow (a, b)\ R\ (e, f)$, since $cd \neq 0$ (R1) 4

Note: *Award* ***(M0)****(R1) if $cd \neq 0$ is not mentioned.*

(b) $ad = bc \Leftrightarrow a : b = c : d$ (M1)

i.e. the classes are those pairs (a, b) and (c, d) with $\frac{a}{b} = \frac{c}{d}$
i.e. the elements of those pairs are in the same ratio.
i.e. the elements are on the same line going through the origin. (R1) 2

[6]

16.

(a) $(a, b)R(p, q) \Rightarrow \max(|a|, |b|) = \max(|p|, |q|)$ ***(M1)***
$\max(|p|, |q|) = \max(|a|, |b|) \Rightarrow (p, q)R(a, b)$
$\Rightarrow R$ is symmetric ***(A1)***

$(a, b)R(a, b) \Rightarrow \max(|a|, |b|) = \max(|a|, |b|)$ ***(M1)***
R is reflexive ***(A1)***

$(a, b)R(x, y)$ and $(x, y)R(p, q) \Rightarrow (a, b)R(p, q)$
since $\max(|a|, |b|) = \max(|x|, |y|)$ and $\max(|x|, |y|) = \max(|p|, |q|)$ ***(M1)***
$\Rightarrow \max(|a|, |b|) = \max(|p|, |q|)$
R is transitive. ***(A1)***

$\Rightarrow R$ is an equivalence relation. ***(AG)***

[6 marks]

(b) (i) If $\max(|x|, |y|) = c$
Then $|x| = c$ and $|y| \le c$
$\Rightarrow x = \pm c$ and $-c \le y \le c$ ***(M1)(A1)***
or $|y| = c$ and $|x| \le c$ ***(M1)***
$\Rightarrow y = \pm c$ and $-c \le x \le c$ ***(A1)***

(ii) i.e. Concentric squares with a centre at (0, 0) ***(A1)***

[5 marks]

[11]

17. (a) $\forall\ a \in \mathbb{Z}, a\ R\ a$ (A1)
$\forall\ a, b \in \mathbb{Z}, a\ R\ b \Rightarrow m$ divides $a - b \Rightarrow m$ divides $b - a \Rightarrow b\ R\ a$ (A1)
$\forall\ a, b, c \in \mathbb{Z}, a\ R\ b$ and $b\ R\ c \Rightarrow m$ divides $(a - b)$ and m divides $(b - c)$
m divides $(a - b) + (b - c) \Rightarrow m$ divides $(a - c) \Rightarrow a\ R\ c$ (A1)
Hence R is an equivalence relation. (C1) 4

(b) For any reasonable attempt to explain that the equivalence relation partitions the set. (C2)
For either the list of equivalence classes that partition $\mathbb{Z}$ or an attempt to explain that there are m equivalence classes. (C2) 4

[8]

18. (a) aRa since $a^2 - a^2 = 0 \equiv 0(\text{mod}5)$ (A1)

$aRb \Rightarrow bRa$ since $a^2 - b^2 = 0(\text{mod}5) \Rightarrow b^2 - a^2 \equiv 0(\text{mod}5)$ (A1)

aRb and $bRc \Rightarrow aRc$ since $a^2 - b^2 \equiv 0(\text{mod}5)$ and $b^2 - c^2 \equiv 0(\text{mod}5)$

$\Rightarrow a^2 - c^2 = a^2 - b^2 + b^2 - c^2 \equiv 0(\text{mod}5)$ (A2)

Hence R is an equivalence relation. (AG) 4

(b) (i) It is the set of all the elements b of Y such that bRa.
(or equivalent) (C2)

(ii) {5,10} (A1)
{1,4,6,9} (A1)
{2,3,7,8} (A1) 5

[9]

19. (a) Reflexive: $7^a \equiv 7^a$ (modulo 10) so aRa (A1)

Symmetric: $7^a \equiv 7^b$ (modulo 10) $\Rightarrow 7^b \equiv 7^a$ (modulo 10) so $aRb \Rightarrow bRa$ (A1)

Transitive: Let $7^a \equiv 7^b$ (modulo 10) and $7^b \equiv 7^c$ (modulo 10) (M1)

Then, $7^a = 7^b + 10\lambda$ and $7^b = 7^c + 10\mu$

so $7^a = 7^c + 10(\lambda + \mu)$ so aRb and $bRc \Rightarrow aRc$ (A1) 4

(b) We note that $7^0 = 1$, $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$

The equivalence classes are therefore

0, 4, 8, … (A1)
1, 5, 9, … (A1)
2, 6, 10, … (A1)
3, 7, 11, … (A1) 4

(c) 7^{503} (modulo 10) $\equiv 7^3$ (modulo 10) = 3. (A1) 1

[9]

20. We show that S is a reflexive, symmetric and transitive relation on X.
Since R is at equivalence relation on Y, it is reflexive, symmetric, and transitive.

For all a in X, reflexivity of R implies $h(a)\ Rh(a)$. By the definition of the relation S on X, $a\ S\ a$ for all a in X. Hence, S is reflexive. (R2)

Let $a\ S\ b$. Then $h(a)\ R\ h\ (b)$ holds on Y. Since R is symmetric, $h(b)\ R\ h\ (a)$ which implies $b\ S\ a$. Since this holds for all a, b in X. S is a symmetric relation on X. (R2)

Let $a\ S\ b$ and $b\ S\ c$ for any a, b, c in X. Then $h(a)\ R\ h(b)$ and $h\ (b)\ R\ h\ (c)$.

Since R is a transitive relation, we get $h(a)\ R\ h\ (c)$ (M1)

By definition of the relation S on X, $a\ S\ c$. Thus S is transitive on X. (R1) 6

[6]

21. (a) reflexive: $f R f$, since $f = I f I^{-1}$, where I is the identity function.

symmetric: $f R g \Rightarrow f = h o g o h^{-1}$, where h is a bijective function

$\Rightarrow g = h^{-1} o f o h$ where h is a bijective function

$\Rightarrow g R f$ since h^{-1} is also bijective function

transitive: $f R g$ *and* $gR k \Rightarrow f = h_1 o g o h_1^{-1}$ and $g = h_2 o k o h_2^{-1}$

$\Rightarrow f = h_1\ o\ h_2 o k o h_2^{-1} o h_1^{-1}$

$\Rightarrow f = (h_1 o h_2) o k o (h_1 o h_2)^{-1}$

$\Rightarrow f R k$ (since $h_1 o h_2$ is also bijective function)

(b) $f(x) = 2x$. If we consider the bijective function $h(x) = x+1$, then $h^{-1}(x) = x-1$

We find the related function

$(h o f o h^{-1})(x) = 2(x-1)+1 = 2x-1$

[12]

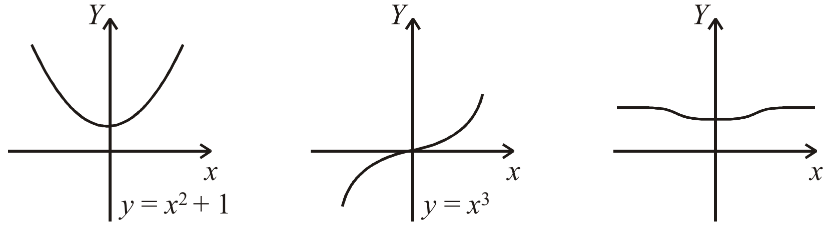
FUNCTIONS

22. $f(n) = f(n')$, for any n, n' in $\mathbb{N}$, implies $n + 1 = n' + 1$.
Hence $n = n'$. Hence f is an injection from $\mathbb{N}$ to $\mathbb{N}$. (R1)

There is no point in the domain of f which is mapped to zero. (M1)
Hence f is not a surjection. (R1) 3

[3]

23. A bijection is both one-to-one and onto, so by considering a sketch of each function



(A1)(A1)(A1)

we can see that for $\mathbb{R}$ to $\mathbb{R}$ only $y = x^3$ is one-to-one and onto. (R1) 4

[4]

24. (a) If the function is injective, then $f(x, y) = f(a, b)$ must imply that $(x, y) = (a, b)$. (R1)

$f(x, y) = f(a, b) \Leftrightarrow (2y - x, x + y) = (2b - a, a + b)$ (M1)
$\Leftrightarrow 2y - x = 2b - a$ and $x + y = a + b \Leftrightarrow 3y = 3b \Leftrightarrow y = b, x = a$ (A1)
$\Leftrightarrow (x, y) = (a, b)$ (R1)

(b) If the function is surjective, then given $(u, v) \in \mathbb{R}^2$, we should be able to find $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (u, v)$. (R1)
$f(x, y) = (u, v) \Leftrightarrow (2y - x, x + y) = (u, v)$ (M1)
$\Leftrightarrow 2y - x = u$ and $x + y = v \Leftrightarrow y = \dfrac{u+v}{3}, x = \dfrac{2v-u}{3}$ (R1)

(c) Since f is injective and surjective, it is bijective. Since every bijective function has an inverse, then f has an inverse. (R1)(R1)

From the last line of the previous part, replace u by x and v by y:

$f^{-1}(x, y) = \left(\dfrac{2y - x}{3}, \dfrac{x + y}{3}\right)$ (A1)

Now $f^{-1}(f(x, y)) = f^{-1}(2y - x, x + y)$ (M1)

$= \left(\dfrac{2(x + y) - (2y - x)}{3}, \dfrac{(2y - x) + (x + y)}{3}\right)$

$= \left(\dfrac{3x}{3}, \dfrac{3x}{3}\right) = (x, y)$ (R1)

[12]

25. (a) (i) f is an increasing function so it is injective. R1 A1
(ii) Let $f(n) = 1$ (or any other appropriate value) M1
Then $5n + 4 = 1$, $n = \dfrac{3}{5}$ which is not in the domain
$\therefore f$ is not surjective. A1

(b) $g(x, y) = (x + 2y, 3x - 5y)$

(i) Let $g(x, y) = g(s, t)$ so $(x + 2y, 3x - 5y) = (s + 2t, 3s - 5t)$ M1
$x + 2y = s + 2t$, $3x - 5y = 3s - 5t$ M1
$y = t$ and $x = s \Rightarrow (x, y) = (s, t)$ g is injective. A1

(ii) Let (u, v) be an element of the codomain.

$x + 2y = u,\ 3x - 5y = v$ M1

Then $-11y = -3u + v$ so $y = \dfrac{3u - v}{11}$ A1

and $11x = 5u + 2v$ so $x = \dfrac{5u - 2v}{11}$ A1

Since $\left(\dfrac{5u+2v}{11}, \dfrac{3u-v}{11}\right)$ is in the domain then g is surjective R1.

(c) $g^{-1}(x, y) = \left(\dfrac{5x+2y}{11}, \dfrac{3x-y}{11}\right)$ (A2) 13

[13]

26.

(a) We need to show that f is surjective and injective. *(R1)*

It is surjective, all elements of S are images. *(R1)*

It is injective, 1:1 function. *(R1)*

So f is a bijection. *(AG)*

[3 marks]

(b) **EITHER**

$f \circ f(1) = 4,\ f \circ f(2) = 3,\ f \circ f(3) = 2,\ f \circ f(4) = 1$ *(A1)*

Therefore, reversing,

$(f \circ f)^{-1}(4) = 1,\ (f \circ f)^{-1}(3) = 2,\ (f \circ f)^{-1}(2) = 3,\ (f \circ f)^{-1}(1) = 4.$ *(A1)*

So, $(f \circ f)^{-1}(x) = (f \circ f)(x)$ for all $x \in S$ *(R1)*

OR

$(f \circ f)x = 4x$ (modulo 5) *(M1)*

So, $(f \circ f) \circ (f \circ f)(x) = 16x$ (modulo 5)

$= x$ (modulo 5) *(A1)*

So, $(f \circ f)(x) = (f \circ f)^{-1}(x)$ for all $x \in S$ *(R1)*

[3 marks]

THEORY - PROOFS

27. There is $\dbinom{n}{0}$ empty subset. (A1)

There are $\dbinom{n}{1}$ subsets with 1 element. (A1)

There are $\dbinom{n}{2}$ subsets with 2 elements.

…………

There are $\dbinom{n}{k}$ subsets with k elements. (A1)

So in total there are $\dbinom{n}{0} + \dbinom{n}{1} + \ldots + \dbinom{n}{n}$ (M1)(A1)

$= (1 + 1)^n = 2^n$ subsets. (A1)(AG)

OR

Since each of the n elements in set X can be either included in the subset or not, there are 2^n possible subsets. (A6)

[6]

28. 29. 30. Answers can be found in the lecture notes.