SETS

1. Let $A$ and $B$ be two non-empty sets, and $A - B$ be the set of all elements of $A$ which are not in $B$. Draw Venn diagrams for $A - B$ and $B - A$ and determine if $B \cap (A - B) = B \cap (B - A)$.

(Total 3 marks)

2. (a) Use a Venn diagram to show that $(A \cup B)' = A' \cap B'$.
(b) Prove that $[(A' \cup B) \cap (A \cup B')]' = (A \cap B)' \cap (A \cup B)$.

(Total 6 marks)

3. The difference, $A - B$, of two sets $A$ and $B$ is defined as the set of all elements of $A$ which do not belong to $B$.
(a) Show by means of a Venn diagram that $A - B = A \cap B'$.
(b) Using set algebra, prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

(Total 5 marks)

4. $A - B$ is the set of all elements that belong to $A$ but not to $B$.
(a) Use Venn diagrams to verify that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.
(b) Use De Morgan’s laws to prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

(Total 6 marks)

5. Use Venn diagrams to show that
(a) $A \cup (B \cap A')' = A \cup B'$
(b) $((A \cap B)' \cup B)' = \emptyset$.

(Total 4 marks)

6. Using de Morgan’s laws, prove that $A \Delta B = (A \cup B) \cap (A \cap B)'$.

(Total 6 marks)

7. Let $A$, $B$ and $C$ be subsets of a given universal set.
(a) Use a Venn diagram to show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
(b) Hence, and by using De Morgan’s laws, show that $(A' \cap B') \cup C' = (A \cap C)' \cap (B \cap C)'$.

(Total 5 marks)

8. Let $X$ and $Y$ be two non-empty sets. Define the operation $X \bullet Y$ by $X \bullet Y = (X \cap Y) \cup (X' \cap Y')$. Prove that $(X \bullet Y)' = (X \cup Y) \cap (X' \cup Y')$.

(Total 3 marks)

9. Define the operation $\#$ on the sets $A$ and $B$ by $A \# B = A' \cup B'$. Show algebraically that
(a) $A \# A = A'$; (b) $(A \# A) \# (B \# B) = A \cup B$; (c) $(A \# B) \# (A \# B) = A \cap B$.

(Total 6 marks)
RELATIONS

10. Let \( A = \{a, b, c, d, e, f\} \), and \( R \) be a relation on \( A \) defined by the matrix below.
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
(Note that a '1' in the matrix signifies that the element in the corresponding row is related to the element in the corresponding column, for example \( dRb \) because there is a '1' on the intersection of the \( d \)-row and the \( b \)-column).
(a) Assuming that \( R \) is transitive, verify that \( R \) is an equivalence relation. (3)
(b) Give the partition of \( A \) corresponding to \( R \). (4)
(Total 7 marks)

11. Let \( S = \{\text{integers greater than 1}\} \). The relation \( R \) is defined on \( S \) by
\[ mRn \iff \gcd(m, n) > 1, \quad m, n \in S. \]
(a) Show that \( R \) is reflexive. (1)
(b) Show that \( R \) is symmetric. (2)
(c) Show using a counter example that \( R \) is not transitive. (3)
(Total 6 marks)

12. The relation \( R \) on \( \mathbb{C} \) is defined as follows
\[ z_1Rz_2 \iff |z_1| = |z_2| \quad \text{for } z_1, z_2 \in \mathbb{C}. \]
(a) Show that \( R \) is an equivalence relation on \( \mathbb{C} \). (3)
(b) Describe the equivalence classes under the relation \( R \). (2)
(Total 5 marks)

13. Let \( S = \{(x, y) \mid x, y \in \mathbb{P}\} \), and let \((a, b), (c, d) \in S\). Define the relation \( \Delta \) on \( S \) as follows:
\[ (a, b) \Delta (c, d) \iff a^2 + b^2 = c^2 + d^2 \]
(a) Show that \( \Delta \) is an equivalence relation. (4)
(b) Find all ordered pairs \((x, y)\) where \((x, y) \Delta (1, 2)\). (2)
(c) Describe the partition created by this relation on the \((x, y)\) plane. (1)
(Total 7 marks)

14. The relation \( R \) is defined on the points \( P(x, y) \) in the plane by
\[ (x_1, y_1)R(x_2, y_2) \iff x_1 + y_2 = x_2 + y_1. \]
(a) Show that \( R \) is an equivalence relation. (4)
(b) Give a geometric description on the equivalence classes. (2)
(Total 7 marks)

15. Consider the set \( \mathbb{Z} \times \mathbb{Z}^+ \). Let \( R \) be the relation defined by the following:
for \((a, b)\) and \((c, d)\) in \( \mathbb{Z} \times \mathbb{Z}^+ \) \[ (a, b)R(c, d) \text{ if and only if } ad = bc, \]
where \( ab \) is the product of the two numbers \( a \) and \( b \).
(a) Prove that \( R \) is an equivalence relation on \( \mathbb{Z} \times \mathbb{Z}^+ \). (4)
(b) Show how \( R \) partitions \( \mathbb{Z} \times \mathbb{Z}^+ \), and describe the equivalence classes. (2)
(Total 6 marks)
16. Let \( \max(|x|, |y|) \) be equal to the largest of \(|x|\) and \(|y|\). Define the relation \( R \) on the \( xy \) plane by

\[(a, b) R (p, q) \iff \max(|a|, |b|) = \max(|p|, |q|) \]

(a) Show that the relation \( R \) is an equivalence relation. \( \text{(6 marks)} \)

(b) (i) Find the equivalence classes.
(ii) Hence describe the equivalence classes. \( \text{(5 marks)} \)

17. Let \( R \) be a relation on \( \mathbb{Z} \) such that for \( m \in \mathbb{Z}^+ \), \( x R y \) if and only if \( m \) divides \( x - y \), where \( x, y \in \mathbb{Z} \).

(a) Prove that \( R \) is an equivalence relation on \( \mathbb{Z} \). \( \text{(4 marks)} \)

(b) Prove that this equivalence relation partitions \( \mathbb{Z} \) into \( m \) distinct classes. \( \text{(4 marks)} \)

18. Let \( Y \) be the set \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.
Define the relation \( R \) on \( Y \) by \( a R b \iff a^2 - b^2 \equiv 0 \pmod{5} \), where \( a, b \in Y \).

(a) Show that \( R \) is an equivalence relation. \( \text{(4 marks)} \)

(b) (i) What is meant by “the equivalence class containing \( a \)?”
(ii) Write down all the equivalence classes. \( \text{(5 marks)} \)

19. The relation \( R \) is defined on the non-negative integers \( a, b \) such that

\[ a R b \iff 7^a \equiv 7^b \pmod{10} \]

(a) Show that \( R \) is an equivalence relation. \( \text{(4 marks)} \)

(b) By considering powers of 7, identify the equivalence classes. \( \text{(4 marks)} \)

(c) Find the value of \( 7^{503} \) (modulo 10). \( \text{(1 mark)} \)

18. Let \( X \) and \( Y \) be two non-empty sets and \( h : X \to Y \),
Let also \( R \) be an equivalence relation on \( Y \).

\( y_1 R y_2 \) denotes that two elements \( y_1 \) and \( y_2 \) of \( Y \) are related.

Define a relation \( S \) on \( X \) by the following:

For all \( a, b \in X \), \( a S b \) if and only if \( h(a) R h(b) \).

Determine if \( S \) is an equivalence relation on \( X \). \( \text{(Total 6 marks)} \)

Notice: the following is not a past paper question; however, it is a modification of a similar past paper question on matrices (here we use bijective functions instead of matrices)

21. Let \( R \) be a relation defined on bijective functions from \( \mathbb{R} \) to \( \mathbb{R} \), given the functions \( f \) and \( g \).

\[ f R g \iff \text{there exists a bijective function } h \text{ such that } f = h \circ g \circ h^{-1} \]

(a) Show that \( R \) is an equivalence relation. \( \text{(a mark)} \)

(b) Find a bijective function related to the function \( f(x) = 2x \).
FUNCTIONS

22. Let \( f : \mathbb{N} \to \mathbb{N} \) be defined by \( f(n) = n + 1 \), for all \( n \in \mathbb{N} \).
Determine if \( f \) is an injection, a surjection, or a bijection. Give reasons for your answer.

23. Determine with reasons which of the following functions is a bijection from \( \mathbb{R} \) to \( \mathbb{R} \).
\[ p(x) = x^2 + 1, \quad q(x) = x^3, \quad r(x) = \frac{x^2 + 1}{x^2 + 2} \]
(Total 4 marks)

24. Define the function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( f(x, y) = (2y - x, x + y) \)
(a) Show that \( f \) is injective.
(b) Show that \( f \) is surjective.
(c) Show that \( f \) has an inverse function. Find this inverse and verify your result.
(Total 12 marks)

25. Consider the functions \( f \) and \( g \), defined by
\[ f : \mathbb{Z} \to \mathbb{Z} \text{ where } f(n) = 5n + 4, \]
\[ g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \text{ where } g(x, y) = (x + 2y, 3x - 5y) \]
(a) Explain whether the function \( f \) is (i) injective; (ii) surjective.
(b) Explain whether the function \( g \) is (i) injective; (ii) surjective.
(c) Find the inverse of \( g \).
(Total 13 marks)

26. Let \( S = \{1, 2, 3, 4\} \) and \( f \) be a function, with domain and range \( S \), defined by
\[ f(x) = 2x \text{ (modulo 5)} \]
(a) Prove the \( f \) is a bijection.
(b) Show that the composite function \( f \circ f \) is its own inverse.
(Total 6 marks)

THEORY – PROOFS

27. Let \( X \) be a set containing \( n \) elements (where \( n \) is a positive integer).
Show that the set of all subsets of \( X \) contains \( 2^n \) elements.

28. Consider any functions \( f : A \to B \) and \( g : B \to C \). Show that
(a) if both \( f \) and \( g \) are injective then \( g \circ f \) is also injective.
(b) if both \( f \) and \( g \) are surjective then \( g \circ f \) is also surjective.
(c) if both \( f \) and \( g \) are bijective then \( g \circ f \) is also bijective.

29. Consider any functions \( f : A \to B \) and \( g : B \to C \).
(a) Given that is \( g \circ f \) injective, show that \( f \) is injective.
(b) Given that is \( g \circ f \) surjective, show that \( g \) is surjective.
(c) Given that is \( g \circ f \) bijective, write down the conclusion for \( f \) and \( g \).

30. Give the definitions of the following terms:

<table>
<thead>
<tr>
<th>Difference: ( A - B )</th>
<th>Symmetric difference: ( A \Delta B )</th>
<th>Cartesian Product ( A \times B )</th>
<th>Relation from ( A ) to ( B )</th>
<th>Relation on ( A )</th>
<th>Equivalence class of ( a ): ([a])</th>
<th>Injection</th>
<th>Surjection</th>
<th>Bijection</th>
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<tbody>
<tr>
<td>Give examples.</td>
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