MATH HL

OPTION - REVISION

SETS, RELATIONS AND GROUPS

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PART A: SETS AND RELATIONS

SETS

1. Let A and B be two non-empty sets, and $A - B$ be the set of all elements of A which are not in B. Draw Venn diagrams for $A - B$ and $B - A$ and determine if $B \cap (A - B) = B \cap (B - A)$.

(Total 3 marks)

2. (a) Use a Venn diagram to show that $(A \cup B)' = A' \cap B'$. **(2)**

(b) Prove that $[(A' \cup B) \cap (A \cup B')]' = (A \cap B)' \cap (A \cup B)$. **(4)**

(Total 6 marks)

3. The difference, $A - B$, of two sets A and B is defined as the set of all elements of A which do not belong to B.

(a) Show by means of a Venn diagram that $A - B = A \cap B'$. **(1)**

(b) Using set algebra, prove that $A - (B \cup C) = (A - B) \cap (A - C)$. **(4)**

(Total 5 marks)

4. $A–B$ is the set of all elements that belong to A but not to B.

(a) Use Venn diagrams to verify that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. **(2)**

(b) Use De Morgan's laws to prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. **(4)**

(Total 6 marks)

5. Use Venn diagrams to show that

(a) $A \cup (B \cap A')' = A \cup B'$ (b) $\left((A \cap B)' \cup B\right)' = \varnothing$.

(Total 4 marks)

6. Using de Morgan's laws, prove that $A \,\Delta\, B = (A \cup B) \cap (A \cap B)'$.

(Total 6 marks)

7. Let A, B and C be subsets of a given universal set.

(a) Use a Venn diagram to show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. **(2)**

(b) Hence, and by using De Morgan's laws, show that

$$(A' \cap B) \cup C' = (A \cap C)' \cap (B' \cap C)'.$$ **(3)**

(Total 5 marks)

8. Let X and Y be two non-empty sets. Define the operation $X \bullet Y$ by $X \bullet Y = (X \cap Y) \cup (X' \cap Y')$. Prove that $(X \bullet Y)' = (X \cup Y) \cap (X' \cup Y')$.

(Total 3 marks)

9. Define the operation # on the sets A and B by $A \# B = A' \cup B'$. Show algebraically that

(a) $A\#A = A'$;   (b) $(A\#A)\#(B\#B) = A \cup B$;   (c) $(A\#B)\#(A\#B) = A \cap B$.

(Total 6 marks)

RELATIONS

10. Let $A = \{a , b , c , d, e , f\}$, and R be a relation on A defined by the matrix below.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Note that a '1' in the matrix signifies that the element in the corresponding row is related to the element in the corresponding column, for example dRb because there is a '1' on the intersection of the d-row and the b-column).

(a) Assuming that R is transitive, verify that R is an equivalence relation. **(3)**

(b) Give the partition of A corresponding to R. **(4)**

(Total 7 marks)

11. Let S = {integers greater than 1}. The relation R is defined on S by

$$m \, R \, n \Leftrightarrow \gcd(m, n) > 1, \text{ for } m, n \in S.$$

(a) Show that R is reflexive. **(1)**

(b) Show that R is symmetric. **(2)**

(c) Show using a counter example that R is not transitive. **(3)**

(Total 6 marks)

12. The relation R on $\mathbb{C}$ is defined as follows

$$z_1 \, R \, z_2 \Leftrightarrow |z_1| = |z_2| \text{ for } z_1, z_2 \in \mathbb{C}.$$

(a) Show that R is an equivalence relation on $\mathbb{C}$. **(3)**

(b) Describe the equivalence classes under the relation R. **(2)**

(Total 5 marks)

13. Let $S = \{(x, y) \mid x, y \in \mathbb{R}\}$, and let $(a, b), (c, d) \in S$. Define the relation Δ on S as follows:

$$(a, b) \, \Delta \, (c, d) \Leftrightarrow a^2 + b^2 = c^2 + d^2$$

(a) Show that Δ is an equivalence relation. **(4)**

(b) Find all ordered pairs (x, y) where $(x, y) \, \Delta \, (1, 2)$. **(2)**

(c) Describe the partition created by this relation on the (x, y) plane. **(1)**

(Total 7 marks)

14. The relation R is defined on the points $P(x, y)$ in the plane by

$$(x_1, y_1) R (x_2, y_2) \text{ if and only if } x_1 + y_2 = x_2 + y_1.$$

(a) Show that R is an equivalence relation. **(4)**

(b) Give a geometric description on the equivalence classes. **(2)**

(Total 7 marks)

15. Consider the set $\mathbb{Z} \times \mathbb{Z}^+$. Let R be the relation defined by the following:

for (a, b) and (c, d) in $\mathbb{Z} \times \mathbb{Z}^+$ $\quad (a, b) \, R \, (c, d)$ if and only if $ad = bc$,

where ab is the product of the two numbers a and b.

(a) Prove that R is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$. (4)

(b) Show how R partitions $\mathbb{Z} \times \mathbb{Z}^+$, and describe the equivalence classes. (2)

(Total 6 marks)

16. Let $\max(|x|, |y|)$ be equal to the largest of $|x|$ and $|y|$. Define the relation R on the xy plane by

$$(a,b)R(p,q) \Leftrightarrow \max(|a|, |b|) = \max(|p|, |q|)$$

(a) Show that the relation R is an equivalence relation. **(6)**

(b) (i) Find the equivalence classes.

(ii) Hence describe the equivalence classes. **(5)**

(Total 11 marks)

17. Let R be a relation on $\mathbb{Z}$ such that for $m \in \mathbb{Z}^+$, $x\,R\,y$ if and only if m divides $x - y$, where $x, y \in \mathbb{Z}$.

(a) Prove that R is an equivalence relation on $\mathbb{Z}$. **(4)**

(b) Prove that this equivalence relation partitions $\mathbb{Z}$ into m distinct classes. **(4)**

(Total 8 marks)

18. Let Y be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Define the relation R on Y by $aRb \Leftrightarrow a^2 - b^2 \equiv 0 \pmod 5$, where $a, b \in Y$.

(a) Show that R is an equivalence relation. **(4)**

(b) (i) What is meant by "the equivalence class containing a"?

(ii) Write down all the equivalence classes. **(5)**

(Total 9 marks)

19. The relation R is defined on the non-negative integers a, b such that

$$aRb \quad \text{if and only if} \quad 7^a \equiv 7^b \text{ (modulo 10)}.$$

(a) Show that R is an equivalence relation. **(4)**

(b) By considering powers of 7, identify the equivalence classes. **(4)**

(c) Find the value of 7^{503} (modulo 10). **(1)**

(Total 9 marks)

20. Let X and Y be two non-empty sets and $h : X \to Y$,
Let also R be an equivalence relation on Y.

$y_1 R y_2$ denotes that two elements y_1 and y_2 of Y are related.

Define a relation S on X by the following:

For all $a, b \in X$, $a\,S\,b$ if and only if $h(a)\,R\,h(b)$.

Determine if S is an equivalence relation on X.

(Total 6 marks)

Notice: the following is not a past paper question; however, it is a modification of a similar past paper question on matrices (here we use bijective functions instead of matrices)

21. Let R be a relation defined on bijective functions from $\mathbb{R}$ to $\mathbb{R}$, given the functions f and g,

$f\,R\,g$ if and only if there exists a bijective function h such that $f = h \circ g \circ h^{-1}$

(a) Show that R is an equivalence relation.

(b) Find a bijective function related to the function $f(x) = 2x$

FUNCTIONS

22. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 1$, for all $n \in \mathbb{N}$.
Determine if f is an injection, a surjection, or a bijection. Give reasons for your answer.

(3)

23. Determine with reasons which of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

$$p(x) = x^2 + 1, \qquad q(x) = x^3, \qquad r(x) = \frac{x^2 + 1}{x^2 + 2}$$

(Total 4 marks)

24. Define the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x, y) = (2y - x, x + y)$

(a) Show that f is injective. **(4)**
(b) Show that f is surjective. **(3)**
(c) Show that f has an inverse function. Find this inverse and verify your result. **(5)**

(Total 12 marks)

25. Consider the functions f and g, defined by

$$f : \mathbb{Z} \to \mathbb{Z} \text{ where } f(n) = 5n + 4,$$

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \text{ where } g(x, y) = (x + 2y, 3x - 5y)$$

(a) Explain whether the function f is (i) injective; (ii) surjective.
(b) Explain whether the function g is (i) injective; (ii) surjective.
(c) Find the inverse of g.

(Total 13 marks)

26. Let $S = \{1,2,3,4\}$ and f be a function, with domain and range S, defined by

$$f(x) = 2x \text{ (modulo 5).}$$

(a) Prove the f is a bijection. (3)
(b) Show that the composite function $f \circ f$ is its own inverse. (3)

(Total 6 marks)

THEORY – PROOFS

27. Let X be a set containing n elements (where n is a positive integer).
Show that the set of all subsets of X contains 2^n elements.

28. Consider any functions $f : A \to B$ and $g : B \to C$. Show that
(a) if both f and g are injective then $g \circ f$ is also injective.
(b) if both f and g are surjective then $g \circ f$ is also surjective.
(c) if both f and g are bijective then $g \circ f$ is also bijective.

29. Consider any functions $f : A \to B$ and $g : B \to C$.
(a) Given that is $g \circ f$ injective, show that f is injective.
(b) Given that is $g \circ f$ surjective, show that g is surjective.
(c) Given that is $g \circ f$ bijective, write down the conclusion for f and g.

30. Give the definitions of the following terms:

Difference: $A - B$ Symmetric difference: $A \Delta B$ Cartesian Product $A \times B$	Relation from A to B Relation on A Equivalence relation	Equivalence class of a: $[a]$ Partition of A	Injection Surjection Bijection

Give examples.