International Baccalaureate

LECTURE NOTES

for FURTHER MATHEMATICS

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TOPIC: LINEAR ALGEBRA
MATRICES

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1. DEFINITION OF A MATRIX – MATRIX OPERATIONS

- A **matrix** is simply a rectangular array of numbers. Some typical examples:

$$\begin{pmatrix} 1 & 2 & 7 \\ -3 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 4 & 6 \\ 2 & 11 \end{pmatrix} \qquad \begin{pmatrix} 4 & -8 \\ 2.5 & 4 \\ 0 & 1/2 \\ 1 & 3.1 \end{pmatrix}$$

A matrix is usually denoted by a capital letter, say

$$A=\begin{pmatrix} 1 & 2 & 7 \\ -3 & 0 & 5 \end{pmatrix} \qquad B=\begin{pmatrix} 4 & 6 \\ 2 & 11 \end{pmatrix} \qquad C=\begin{pmatrix} 4 & -8 \\ 2.5 & 4 \\ 0 & 1/2 \\ 1 & 3.1 \end{pmatrix}$$

The first matrix above has **2 rows** and **3 columns**.

We say that A is a **2x3** ("two by three") matrix, or otherwise that **the order** of A is **2x3**.

Likewise, B is a **2x2** matrix while C is a **4x2** matrix.

The general form of a 2x3 matrix is

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

(notice that a_{23} for example is the element in row 2 and column 3)

- SQUARE MATRICES

No of rows = No of columns

The order of a square matrix is nxn, eg. 2x2, 3x3, 4x4 etc.

(in this case we may also say: "a square matrix of order n".

For example,

$$\begin{pmatrix} 4 & 6 \\ 5 & 2 \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

are square matrices of order 2 and 3 respectively.

We also say that the elements a_{11}, a_{22}, a_{33} … (indicated above) form the **main diagonal** of the square matrix.

♦ ROW MATRICES

Matrices of order 1xn

For example

$A=(2 \quad 7 \quad 3)$ is a 1x3 matrix

$B=(4 \quad 2 \quad 6 \quad -1 \quad 0 \quad 6)$ is a 1x6 matrix

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♦ COLUMN MATRICES

Matrices of order mx1

For example

$A=\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ is a 3x1 matrix $\qquad B=\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is a 2x1 matrix

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Notice: Matrices of order 1x1 are also defined, for example C=(5).

♦ THE ZERO MATRIX **O**

All elements are 0

The 2x3 **zero matrix** is $\quad \mathbf{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The 2x2 **zero matrix** is $\quad \mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so on!

- EQUAL MATRICES: A=B

A=B if	• A and B have the same order • the corresponding elements are equal

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EXAMPLE 1

$$A=\begin{pmatrix}2 & x\\ a & 3\end{pmatrix}\quad B=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 0\end{pmatrix}\quad C=\begin{pmatrix}y & 7\\ 0 & 3\end{pmatrix}\quad D=\begin{pmatrix}s & t & u\\ v & w & 7\end{pmatrix}$$

It cannot be A=B since A is 2x2 while B is 2x3

A=C implies y=2, x=7, a=0

(3=3 holds anyway!)

$B\neq D$ since $0\neq 7$ (although both are 2x3)

Let A and B have the same order. We define some new matrices:

- THE SUM A+B

we simply add the corresponding elements

- THE DIFFERENCE A-B

we simply subtract the corresponding elements

- THE SCALAR PRODUCT nA (n is a scalar, i.e. a number)

we simply multiply each element of A by n

EXAMPLE 2

$$A=\begin{pmatrix}1 & 2 & 3\\ 4 & 1 & 5\end{pmatrix} \qquad B=\begin{pmatrix}3 & 5 & 0\\ 2 & 3 & 7\end{pmatrix}$$

Then

$$A+B=\begin{pmatrix}4 & 7 & 3\\ 6 & 4 & 12\end{pmatrix} \qquad A-B=\begin{pmatrix}-2 & -3 & 3\\ 2 & -2 & -2\end{pmatrix} \qquad B-A=\begin{pmatrix}2 & 3 & -3\\ -2 & 2 & 2\end{pmatrix}$$

$$3A=\begin{pmatrix}3 & 6 & 9\\ 12 & 3 & 15\end{pmatrix} \qquad -3A=\begin{pmatrix}-3 & -6 & -9\\ -12 & -3 & -15\end{pmatrix} \qquad -A=-1A=\begin{pmatrix}-1 & -2 & -3\\ -4 & -1 & -5\end{pmatrix}$$

$$\frac{1}{2}A=\begin{pmatrix}1/2 & 1 & 3/2\\ 2 & 1/2 & 5/2\end{pmatrix} \quad \text{or} \quad 0.5A=\begin{pmatrix}0.5 & 1 & 1.5\\ 1 & 0.5 & 2.5\end{pmatrix}$$

Finally,

$$\begin{aligned}2A+3B &= 2\begin{pmatrix}1 & 2 & 3\\ 4 & 1 & 5\end{pmatrix}+3\begin{pmatrix}3 & 5 & 0\\ 2 & 3 & 7\end{pmatrix}\\ &= \begin{pmatrix}2 & 4 & 6\\ 8 & 2 & 10\end{pmatrix}+\begin{pmatrix}9 & 15 & 0\\ 6 & 9 & 21\end{pmatrix}\\ &= \begin{pmatrix}11 & 19 & 6\\ 14 & 11 & 31\end{pmatrix}\end{aligned}$$

EXAMPLE 3

Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0\\ 5 & -1\end{pmatrix}$. Find the matrix X if

$$2A+X=3B$$

Method 1:

X must be of size 2x2. Suppose that $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Then

$$2A+X=3B \quad \Leftrightarrow 2\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+\begin{pmatrix}a & b\\ c & d\end{pmatrix}=3\begin{pmatrix}2 & 0\\ 5 & -1\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 15 & -3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2+a & 4+b \\ 6+c & 8+d \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 15 & -3 \end{pmatrix}$$

Hence,

$2+a=6 \Leftrightarrow \boxed{a=4}$ $\quad$ $4+b=0 \Leftrightarrow \boxed{b=-4}$

$6+c=15 \Leftrightarrow \boxed{c=9}$ $\quad$ $8+d=-3 \Leftrightarrow \boxed{d=-11}$

Therefore,

$$\boxed{X=\begin{pmatrix} 4 & -4 \\ 9 & -11 \end{pmatrix}}$$

<u>Method 2:</u>

We solve for X (as a usual equation)

$$2A+X=3B \quad \Leftrightarrow \quad X=3B-2A$$

$$\Leftrightarrow \quad X=3\begin{pmatrix} 2 & 0 \\ 5 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\Leftrightarrow \quad X=\begin{pmatrix} 6 & 0 \\ 15 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

$$\Leftrightarrow \quad \boxed{X=\begin{pmatrix} 4 & -4 \\ 9 & -11 \end{pmatrix}}$$

The following operation is the most important one for matrices

- THE PRODUCT OF TWO MATRICES: AB

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} = ?$$

It is NOT $\begin{pmatrix} 2 & 6 \\ 15 & 4 \end{pmatrix}$ as someone would expect!! Here, we do not multiply the corresponding elements.

Let us start by a SPECIAL CASE. We multiply a row-matrix by a column-matrix:

$$\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (ax+by+cz)$$

Notice that we multiply the corresponding elements and add the results. For example,

$$\begin{pmatrix} 2 & 3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = (2 \cdot 3+3 \cdot 4+5 \cdot 1)=(23)$$

Now we are ready to multiply two matrices A and B in general. First of all, the orders of A and B must be as follows:

A	B
m x k	k x n

That is,

number of columns of A = number of rows of B.

The order of the new matrix AB will be **m x n**

For example,

order of A	order of B	order of AB
3x5	5x8	3x8
3x2	2x1	3x1
2x2	2x2	2x2
2x3	2x3	not defined

The multiplication takes place as follows:

- we multiply rows of A by columns of B
- (row i) x (column j) will give the element a_{ij}

Ok, it seems complicated!!! Relax, take it easy!!!! Look at the following description:

Let

$$A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} \qquad B=\begin{pmatrix}a & x\\ b & y\\ c & z\end{pmatrix}$$

Notice that the order of AB is expected to be 2x2

- For the first row of AB:

  multiply **row 1** of A by each column of B separately:

$$AB=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}a & x\\ b & y\\ c & z\end{pmatrix}=\begin{pmatrix}1a+2b+3c & *\\ * & *\end{pmatrix}$$

$$=\begin{pmatrix}1a+2b+3c & 1x+2y+3z\\ * & *\end{pmatrix}$$

- For the second row of AB:

  multiply **row 2** of A by each column of B separately:

$$AB=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}a & x\\ b & y\\ c & z\end{pmatrix}=\begin{pmatrix}1a+2b+3c & 1x+2y+3z\\ 4a+5b+5c & *\end{pmatrix}$$

$$=\begin{pmatrix}1a+2b+3c & 1x+2y+3z\\ 4a+5b+5c & 4x+5y+6z\end{pmatrix}$$

EXAMPLE 4

$$A=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} \qquad B=\begin{pmatrix}2 & 5\\ 3 & 3\\ 1 & 2\end{pmatrix}$$

Then

$$AB=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}\begin{pmatrix}2 & 5\\ 3 & 3\\ 1 & 2\end{pmatrix}=\begin{pmatrix}11 & 17\\ & \end{pmatrix}$$

since $1\cdot2+2\cdot3+3\cdot1=11$
$1\cdot5+2\cdot3+3\cdot2=17$

$$=\begin{pmatrix}11 & 17\\ 29 & 47\end{pmatrix}$$

since $4\cdot2+5\cdot3+6\cdot1=29$
$4\cdot5+5\cdot3+6\cdot2=47$

EXAMPLE 5

Consider again the matrices A and B above. Let us find the product BA. Notice that the order is expected to be 3x3.

We obtain

$$BA=\begin{pmatrix}2 & 5\\ 3 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}=\begin{pmatrix}2+20 & 4+25 & 6+30\\ 3+12 & 6+15 & 9+18\\ 1+8 & 2+10 & 3+12\end{pmatrix}$$

$$=\begin{pmatrix}22 & 29 & 36\\ 15 & 21 & 27\\ 9 & 12 & 15\end{pmatrix}$$

NOTICE: In general

$$AB \neq BA$$

They may be of different order, or perhaps only one the products could be defined (eg. if A is a 2x3 and B is 3x5)

Even if both matrices A and B are square matrices, say 2x2, the resulting 2x2 matrices AB and BA are not equal in general.

EXAMPLE 6

Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$. Then

$$AB=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}=\begin{pmatrix}1 & 5\\ 3 & 11\end{pmatrix} \quad \text{while} \quad BA=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}4 & 6\\ 6 & 8\end{pmatrix}$$

Sometimes though, it happens AB=BA. Then we say that matrices A and B **commute**.

EXAMPLE 7

Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}5 & 4\\ 6 & 11\end{pmatrix}$. Then

$$AB=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 4\\ 6 & 11\end{pmatrix}=\begin{pmatrix}17 & 26\\ 39 & 56\end{pmatrix} \text{ and } BA=\begin{pmatrix}5 & 4\\ 6 & 11\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}17 & 26\\ 39 & 56\end{pmatrix}$$

The following square matrix plays a key role in our theory.

- THE IDENTITY MATRIX **I**

1 in the main diagonal
0 elsewhere

Namely,

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ or } I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \text{ or } I=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \text{ etc}$$

EXAMPLE 8

Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Then

$$AI=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=A \quad \text{and} \quad IA=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=A$$

In general, for any matrix A

$$AI=A \quad \text{and} \quad IA=A$$

(provided that the orders of A and I are appropriate!)

In other words, the identity matrix I plays the role of 1 (unit) when we multiply matrices!

2. THE DETERMINANT detA – THE INVERSE A^{-1}

♦ 2x2 DETERMINANT

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\det A = ad-bc$$

It is also denoted by

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=ad-bc \qquad \text{or} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$$

EXAMPLE 1

Let $A=\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$. Then $\det A=\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}=5\cdot4-2\cdot3=14$

Let $B=\begin{pmatrix} 2 & -4 \\ 3 & -6 \end{pmatrix}$. Then $\det B=\begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix}=-12+12=0$

EXAMPLE 2

Solve the equation

$$\begin{vmatrix} x & -1 \\ 2 & x-3 \end{vmatrix}=0$$

It is

$$x(x-3)+2=0 \quad \Leftrightarrow \quad x^2-3x+2=0 \quad \Leftrightarrow \quad x=1 \text{ or } x=2$$

♦ 3x3 DETERMINANT

Let $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$. The **determinant** of A is the number

$$\det A = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

Ok, I know, it looks horrible!!!!

A more elegant way to estimate detA is

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

NOTICE:

- We multiply the elements of row 1, a_1,a_2,a_3 by three little determinants respectively.
- For a_1, the corresponding 2x2 determinant can be obtained if we eliminate the row and the column of a_1

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

  Similarly for a_2 and a_3.

- Notice also that the signs alternate.

EXAMPLE 3

Let A= $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 2 & 8 \end{pmatrix}$. Then

$$\det A = 2\begin{vmatrix} 6 & 7 \\ 2 & 8 \end{vmatrix} -3\begin{vmatrix} 5 & 7 \\ 1 & 8 \end{vmatrix} +4\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}$$

$$= 2\cdot 34 \quad - 3\cdot 33 \quad + 4\cdot 4$$

$$= -15$$

The following terminology is also used

If detA=0 we say that the matrix A is **singular**.

If detA≠0 we say that the matrix A is **non-singular**.

EXAMPLE 4

Let $A=\begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix}$ $B=\begin{pmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{pmatrix}$ $C=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

Matrices like A are known us *upper-triangular*

Matrices like B are known us lower-triangular

Matrices like C are known us diagonal

We can easily verify that

$\det A = a\cdot b\cdot c$ $\det B = a\cdot b\cdot c$ $\det C = a\cdot b\cdot c$

For example,

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2\cdot 3\cdot 4 = 24 \qquad \begin{vmatrix} 2 & \sqrt{2} & 5 \\ 0 & 3 & -8 \\ 0 & 0 & 4 \end{vmatrix} = 2\cdot 3\cdot 4 = 24$$

- THE INVERSE A^{-1} OF A 2X2 MATRIX

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **inverse** of A is a new matrix given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It is defined only if $\det A \neq 0$.

EXAMPLE 5

Let $A=\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$. Then detA=2 and the inverse matrix is

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -6 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1/2 & 1 \end{pmatrix}$$

Let $B=\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$. Then detB=0 and hence, B^{-1} is not defined.

Let us multiply the matrices A and A^{-1} of the example above.

$$AA^{-1}=\begin{pmatrix}2 & 6\\ 1 & 4\end{pmatrix}\begin{pmatrix}2 & -3\\ -1/2 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

$$A^{-1}A=\begin{pmatrix}2 & -3\\ -1/2 & 1\end{pmatrix}\begin{pmatrix}2 & 6\\ 1 & 4\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

This is not accidental! In general

$$AA^{-1}=I \quad \text{and} \quad A^{-1}A=I$$

NOTICE: Compare with numbers

NUMBERS	MATRICES
The inverse of number a is a^{-1} $aa^{-1}=1$ and $a^{-1}a=1$	The inverse of a matrix A is A^{-1} $AA^{-1}=I$ and $A^{-1}A=I$
Is any number invertible? NO. Only if $a\neq 0$ (if $a=0$, a^{-1} is not defined)	Is any Matrix invertible? NO. Only if $\det A\neq 0$ (if $\det A=0$, A^{-1} is not defined)
If a is invertible then $a^{-1}=\frac{1}{a}$	If A is invertible then $A^{-1}=\frac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

NOTICE

If AB=I or BA=I we know that A is invertible and B is the inverse of A, that is

$$A^{-1} = B$$

(B is also invertible and $B^{-1} = A$).

♦ THE INVERSE A^{-1} OF A 3X3 MATRIX

The explicit formula for A^{-1} is out of our scope! It is enough to know that

- A^{-1} exists only if $\det A \neq 0$
- $AA^{-1}=I$ and $A^{-1}A=I$
- If AB=I or BA=I then B is the inverse of A
- A^{-1} may be found by calculator (GDC)

EXAMPLE 6

Let $A=\begin{pmatrix} 2 & 5 & 1 \\ 3 & 2 & 0 \\ 4 & 3 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 3 & -2 \\ 0 & -4 & 3 \\ 1 & 14 & -11 \end{pmatrix}$ $C=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

a) Find AB and BA

b) Find the inverse of A

c) Find the inverse of B

Solution

a) We can easily see that AB=I and BA=I

b) Clearly $A^{-1}=B$ (the GDC also gives the same result)

c) Similarly $B^{-1}=A$ (the GDC also gives the same result)

Notice that we cannot divide matrices, for example

B/A is not defined.

However, we can multiply B by A^{-1}, either as BA^{-1} or as $A^{-1}B$, according to the situation.

The following example will be characteristic:

EXAMPLE 6

Let A=$\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$ and C=$\begin{pmatrix} 20 & 28 \\ 13 & 18 \end{pmatrix}$. Find B given that AB=C

Unfortunately, we cannot say that B=C/A

Method 1: (analytical)

Let B=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then AB=$\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 2a+6c & 2b+6d \\ a+4c & b+4d \end{pmatrix}$

Then AB=C implies

$$\begin{pmatrix} 2a+6c & 2b+6d \\ a+4c & b+4d \end{pmatrix}=\begin{pmatrix} 20 & 28 \\ 13 & 18 \end{pmatrix}$$

That is,

$$2a+6c=20 \qquad 2b+6d=28$$
$$a+4c=13 \qquad b+4d=18$$

The first two equations give a=1, c=3

The second two equations give b=2, d=4.

Therefore,

$$B=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

A much more practical method is the following

Method 2: (solve for B)

The matrix A is invertible with $A^{-1}=\begin{pmatrix} 2 & -3 \\ -1/2 & 1 \end{pmatrix}$.

Thus, we may multiply both parts of AB=C by A^{-1} on the left

$$AB=C \Leftrightarrow A^{-1}AB=A^{-1}C \Leftrightarrow IB=A^{-1}C \Leftrightarrow$$

$$B=A^{-1}C=\begin{pmatrix} 2 & -3 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 20 & 28 \\ 13 & 18 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

♦ EQUATIONS WITH MATRICES

Suppose that square matrices A,B,C are known.

Find the unknown matrix X in each of the following equations:

Matrix equation	**Solution**
$A+X=B$	$X=B-A$
$2A+X=3B$	$X=3B-2A$
$-3A+2X=5B$	$2X=3A+5B \Leftrightarrow X=1/2(3A+5B)$
$AX=B$	$X=A^{-1}B$ (mind the order!)
$XA=B$	$X=BA^{-1}$ (mind the order!)
$AXB=C$	$X=A^{-1}CB^{-1}$
$AX+B=A$	$AX=A-B \Leftrightarrow X=A^{-1}(A-B) \Leftrightarrow X=I-A^{-1}B$
$XA-B=XC$	$XA-XC=B \Leftrightarrow X(A-C)=B \Leftrightarrow X=B(A-C)^{-1}$
$AX+X=B$	$(A+I)X=B \Leftrightarrow X=(A+I)^{-1}B$

Let us solve in detail some of the above equations:

<u>EXAMPLE 7</u>

Let $A=\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$.

Solve the equations

a) $-3A+2X=5B$

b) $AX=B$

c) $XA=B$

d) $AX+B=A$

e) $AX+X=B$

Solution

a) $-3A+2X=5B \Leftrightarrow X = 1/2(3A+5B)$

$\Leftrightarrow X = 1/2 \begin{pmatrix} 21 & 35 \\ 8 & 19 \end{pmatrix} = \begin{pmatrix} 21/2 & 35/2 \\ 4 & 19/2 \end{pmatrix}$

b) $AX = B \Leftrightarrow X = A^{-1}B$

$\Leftrightarrow X = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

$\Leftrightarrow X = \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix}$

c) $XA = B \Leftrightarrow X = BA^{-1}$

$\Leftrightarrow X = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$\Leftrightarrow X = \begin{pmatrix} 5 & -7 \\ 1 & -1 \end{pmatrix}$

d) $AX+B=A \Leftrightarrow X = I-A^{-1}B$

$\Leftrightarrow X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix}$

$\Leftrightarrow X = \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$

e) $AX+X=B \Leftrightarrow X = (A+I)^{-1}B$

$\Leftrightarrow X = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

$\Leftrightarrow X = \frac{1}{7}\begin{pmatrix} 4 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

$\Leftrightarrow X = \frac{1}{7}\begin{pmatrix} 7 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 6/7 \\ 0 & 2/7 \end{pmatrix}$

3. SYSTEMS OF LINEAR EQUATIONS

♦ THE FORM AX=B

Consider the system

$$2x+3y=9$$
$$4x+7y=19$$

If we solve it in the traditional way (or by GDC) we will realize that x=3 and y=1 (check!)

Let

$A=\begin{pmatrix}2 & 3\\ 4 & 7\end{pmatrix}$ the matrix of coefficients

$X=\begin{pmatrix}x\\ y\end{pmatrix}$ the matrix of unknowns

$B=\begin{pmatrix}9\\ 19\end{pmatrix}$ the matrix of constants

The equation of matrices AX=B is equivalent to

$$\begin{pmatrix}2 & 3\\ 4 & 7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}9\\ 19\end{pmatrix}, \qquad \text{that is} \qquad \begin{pmatrix}2x+3y\\ 4x+7y\end{pmatrix}=\begin{pmatrix}9\\ 19\end{pmatrix}$$

which gives in fact the system of linear equations above!

Consider now the system

$$5x+11y-21z = -22$$
$$x +2y -4z = -4$$
$$3x -2y +3z = 11$$

Again, if

$$A=\begin{pmatrix}5 & 11 & -21\\ 1 & 2 & -4\\ 3 & -2 & 3\end{pmatrix} \qquad X=\begin{pmatrix}x\\ y\\ z\end{pmatrix} \qquad B=\begin{pmatrix}-22\\ -4\\ 11\end{pmatrix}$$

the system can be written in the form AX=B

In general, any system of n linear equations and n unknowns can be expressed in the form

$$AX=B$$

Hence, if A is invertible (that is if $\det A \neq 0$) the solution is given by

$$X=A^{-1}B$$

EXAMPLE 1

Consider the 2x2 system given above

$$2x+3y=9$$
$$4x+7y=19$$

which can be written in the form AX=B.

Since $\det A=\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix}=2\neq 0$,

$$A^{-1}=\frac{1}{2}\begin{pmatrix} 7 & -3 \\ -4 & 2 \end{pmatrix}=\begin{pmatrix} 7/2 & -3/2 \\ -2 & 1 \end{pmatrix}$$

The solution of the system is given by

$$X = A^{-1}B = \begin{pmatrix} 7/2 & -3/2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 19 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words, x=3 and y=1.

EXAMPLE 2

Consider the 3x3 system given above

$$5x+11y-21z = -22$$
$$x +2y -4z = -4$$
$$3x -2y +3z = 11$$

which can be written in the form AX=B.

Find the solution of the system, given that

$$A^{-1} = \begin{pmatrix} 2/7 & -9/7 & 2/7 \\ 15/7 & -78/7 & 1/7 \\ 8/7 & -43/7 & 1/7 \end{pmatrix}$$

The solution of the system is given by

$$X = A^{-1}B = \begin{pmatrix} 2/7 & -9/7 & 2/7 \\ 15/7 & -78/7 & 1/7 \\ 8/7 & -43/7 & 1/7 \end{pmatrix}\begin{pmatrix} -22 \\ -4 \\ 11 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

EXAMPLE 3

Let $A=\begin{pmatrix} 2 & 5 & 1 \\ 3 & 2 & 0 \\ 4 & 3 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 0 & 3 & -2 \\ 0 & -4 & 3 \\ 1 & 14 & -11 \end{pmatrix}$, $C=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $D=\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

d) Find AB

e) Write down the system of three linear equations which corresponds to the matrix equation BX=C

f) Solve the system of linear equations BX=C

g) Solve the matrix equation BX=D (this is not a system)

Solution

a) We easily obtain AB=I. Hence, A and B are inverse to each other.

b)

$$\begin{aligned} 3y - 2z &= 1 \\ -4y + 3z &= 1 \\ x + 14y - 11z &= 1 \end{aligned}$$

c) $BX=C \Leftrightarrow X=B^{-1}C \Leftrightarrow X=AC \Leftrightarrow X=\begin{pmatrix} 2 & 5 & 1 \\ 3 & 2 & 0 \\ 4 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow X=\begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix}$

That is $x=8$, $y=5$, $z=7$

d) $BX=D \Leftrightarrow X=B^{-1}D \Leftrightarrow X=AD \Leftrightarrow X=\begin{pmatrix} 2 & 5 & 1 \\ 3 & 2 & 0 \\ 4 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \Leftrightarrow X=\begin{pmatrix} 8 & 8 \\ 5 & 5 \\ 7 & 7 \end{pmatrix}$

Notice though that if A is not invertible (A^{-1} does not exist) the system AX=B cannot be solved in this way. In general,

for the system AX=B

$\det A \neq 0$ (A^{-1} exists)	• UNIQUE SOLUTION **$X=A^{-1}B$**
$\det A = 0$ (A^{-1} doesn't exist)	• NO SOLUTION, or • INFINITELY MANY (∞) SOLUTIONS

REMARK. Compare with numbers. For the equation **ax=b**

$a \neq 0$ ($a^{-1}=\frac{1}{a}$ exists)	• Unique solution $x=\frac{b}{a}$.
$a=0$ (a^{-1} doesn't exist)	• No solution (e.g. 0x=5 has no solution) • Infinitely many solutions (e.g. 0x=0, true for any $x \in R$)

Let us see two examples of systems of linear equations, where detA=0

EXAMPLE 4

Consider the systems

(a) $x+2y=1$, $2x+4y=5$ (b) $x+2y=1$, $2x+4y=2$

For both $\det A=\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}=0$,

so the systems have either no solution or an ∞ number of solutions.

We multiply the first solution by 2 and obtain 2x+4y=2. Hence the two systems take the equivalent form

(a) 2x+4y=2 (b) 2x+4y=2

2x+4y=5 2x+4y=2

- System (a) has no solution (impossible)
- System (b) reduces to just one equation: x+2y=1.

  There are infinitely many solutions:

  We solve for x:

  $x=1-2y$,

  $y\in R$ (free variable).

[for several values of y we obtain different solutions, for example (1,0), (-1,1), (-3,2), (3,-1), etc]

For 3x3 systems with detA=0, only Math HL needs to go further (next section!)

4. THE AUGMENTED MATRIX

In this section we investigate further the system of linear equations

AX=B

Let us present first an alternative way of getting the unique solution, in case detA≠0

♦ USE OF DETERMINANTS

Consider the system

$$\begin{aligned} a_1x+b_1y&=c_1 \\ a_2x+b_2y&=c_2 \end{aligned}$$

Set

$$D=\det A=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

D_x: constants in the column of x

D_y: constants in the column of y

If D≠0, the unique solution is

$$x=\frac{D_x}{D}, \quad y=\frac{D_y}{D}$$

Proof:

The system can be written in the form AX=B. Notice that

$$A^{-1}=\frac{1}{D}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}.$$

Thus, the unique solution is

$$X= A^{-1}B=\frac{1}{D}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}=\frac{1}{D}\begin{pmatrix} b_2c_1-b_1c_2 \\ -a_2c_1+a_1c_2 \end{pmatrix}=\frac{1}{D}\begin{pmatrix} D_x \\ D_y \end{pmatrix}=\begin{pmatrix} D_x/D \\ D_y/D \end{pmatrix}$$

And hence the result!

EXAMPLE 1

Consider again the 2x2 system given in Example 1 of the previous section:

$$2x+3y=9$$
$$4x+7y=19$$

We have

$$D=\begin{vmatrix}2 & 3\\4 & 7\end{vmatrix}=2\neq 0,\quad D_x=\begin{vmatrix}9 & 3\\19 & 7\end{vmatrix}=6\qquad D_y=\begin{vmatrix}2 & 9\\4 & 19\end{vmatrix}=2$$

Therefore,

$$x=\frac{D_x}{D}=\frac{6}{2}=3,\quad y=\frac{D_y}{D}=\frac{2}{2}=1$$

as expected!

Similarly, for a 3x3 system

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

the determinants D, D_x, D_y, D_z are defined in an analogue way and if $D\neq 0$ the unique solution is

$$x=\frac{D_x}{D},\quad y=\frac{D_y}{D},\quad z=\frac{D_z}{D}$$

EXAMPLE 2

Consider again the 3x3 system given in Example 2 of the previous section

$$5x+11y-21z=-22$$
$$x+2y-4z=-4$$
$$3x-2y+3z=11$$

Then

$$D=\begin{vmatrix} 5 & 11 & -21 \\ 1 & 2 & -4 \\ 3 & -2 & 3 \end{vmatrix}=-7\neq 0$$

$$D_x=\begin{vmatrix} -22 & 11 & -21 \\ -4 & 2 & -4 \\ 11 & -2 & 3 \end{vmatrix}=-14,\quad D_y=\begin{vmatrix} 5 & -22 & -21 \\ 1 & -4 & -4 \\ 3 & 11 & 3 \end{vmatrix}=7,\quad D_z=\begin{vmatrix} 5 & 11 & -22 \\ 1 & 2 & -4 \\ 3 & -2 & 11 \end{vmatrix}=-7$$

and the unique solution is

$$x=\frac{D_x}{D}=2,\quad y=\frac{D_y}{D}=-1,\quad z=\frac{D_z}{D}=1$$

as expected!

Let us collect the information we have until now.

- METHODOLOGY FOR 2X2 AND 3X3 SYSTEMS: AX=B

  1) We find D=detA
  2) If $D\neq 0$ there is a UNIQUE SOLUTION, given by

     $$X=A^{-1}B$$

     or in detail by

     $$x=\frac{D_x}{D},\quad y=\frac{D_y}{D},\quad \dots$$

  3) If D=0 we say that the system has either
     - NO SOLUTION
     - ∞ NUMBER OF SOLUTIONS

     For 2x2 systems, it is easy to investigate the solution (look at Example 4 in previous section)

     For 3x3 systems, we will use the method of the **augmented matrix** described below

---

Consider the system

$$\begin{aligned} x + \ y - 2z &= 4 \\ 2x + 3y + 3z &= 3 \\ 5x + 7y + 4z &= 5 \end{aligned}$$

Here,

$$D = \det A = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & 3 \\ 5 & 7 & 4 \end{vmatrix} = 0$$

Hence, there is either NO SOLUTION or AN INFINITE NUMBER OF SOLUTIONS.

STEP 1: Use equation 1 to eliminate x from equations 2 and 3:

$$\begin{aligned} x + y - 2z &= 4 \\ y + 7z &= -5 && [\mathbf{Equ2} - 2 \times \mathbf{Equ1}] \\ 2y + 14z &= -15 && [\mathbf{Equ3} - 5 \times \mathbf{Equ1}] \end{aligned}$$

STEP 2: Use equation 2 to eliminate x from equation 3:

$$\begin{aligned} x + y - 2z &= 4 \\ y + 7z &= -5 \\ 0 &= -5 && [\mathbf{Equ3} - 5 \times \mathbf{Equ2}] \end{aligned}$$

The 3rd equation implies that there is NO SOLUTION.

In fact, the unknowns x,y,z do not play any role during this process! We can work only with coefficients placed in an **augmented** **matrix** as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 5 \end{array}\right) \begin{array}{ll} R_1 & \text{(row1)} \\ R_2 & \text{(row2)} \\ R_3 & \text{(row3)} \end{array}$$

Then we proceed, step by step, to equivalent augmented matrices by performing appropriate row operations. The equivalence between two matrices is denoted by the symbol ~ :

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -15 \end{array}\right) \begin{array}{l} R_1 \\ R_2-2R_1 \\ R_3-5R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & -5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3-2R_2 \end{array}$$

The last row implies that the system has NO SOLUTION as it corresponds to the equation $0x+0y+0z=-5$

In general, the row operations we may perform in order to obtain equivalent matrices are the following

- Interchange rows (e.g. $R_1 \leftrightarrow R_2$)
- Multiply a row by a scalar (e.g. $R_1 \to 5R_1$)
- Add to a row the multiple of another row (e.g. $R_1 \to R_1 \pm 3R_2$)

♦ METHODOLOGY FOR 3X3 SYSTEMS WITH detA=0

1) We consider the augmented matrix of the system

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

2) We transform to equivalent matrices of the form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right) \quad \text{by using row } R_1 \text{ as a guide}$$

and then

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right) \quad \text{by using row } R_2 \text{ as a guide}$$

If the system has either NO SOLUTION or INFINITELY MANY SOLUTIONS, we expect an equivalent matrix of the form

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & d \end{array}\right)$$

3) a) if $d \neq 0$ the system has NO SOLUTION

b) if $d=0$, the system has INFINITELY MANY SOLUTIONS:

We eliminate the last row of the augmented matrix and transform it into the form

$$\left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right) \quad \text{(by using row } R_2 \text{ as a guide).}$$

The matrix $\left(\begin{array}{ccc|c} 1 & 0 & a & d_1 \\ 0 & 1 & b & d_2 \end{array}\right)$ implies in fact that $\begin{aligned} x+az &= d_1 \\ y+bz &= d_2 \end{aligned}$

We finally say that the solution is

$$x = d_1 - az$$
$$y = d_2 - bz$$
$$z \in R \text{ (free variable)}$$

REMARK

Ideally, in step 2 we attempt to have matrices of the form

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right) \text{ and } \left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right)$$

That is, the leading coefficient of the guide row should be 1.

EXAMPLE 3

Consider the system

$$\begin{aligned} 2x + 3y + 3z &= 3 \\ x + y - 2z &= 4 \\ 5x + 7y + 4z &= 5 \end{aligned}$$

Since $\det A=0$ we use the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & 3 & 3 \\ 1 & 2 & -2 & 4 \\ 5 & 7 & 4 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 5 \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -15 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & -5 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - 2R_2 \end{array}$$

Hence the system has **no solution**.

EXAMPLE 4

Consider the system

$$\begin{aligned} 2x + 3y + 3z &= 3 \\ x + y - 2z &= 4 \\ 5x + 7y + 4z &= 10 \end{aligned}$$

Since detA=0 we use the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & 3 & 3 \\ 1 & 2 & -2 & 4 \\ 5 & 7 & 4 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 7 & 4 & 10 \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 2 & 14 & -10 \end{array}\right) \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - 2R_2 \end{array}$$

Hence the system has **infinitely many solutions**. We carry on by considering

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 0 & 1 & 7 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -16 & 14 \\ 0 & 1 & 7 & -5 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \end{array}$$

Hence,

$$\begin{aligned} x - 16z &= 14 \\ y - 7z &= -5 \end{aligned}$$

and finally

$$\begin{aligned} x &= 14 + 16z \\ y &= -5 + 7z \\ z &\in R \text{ (free variable)} \end{aligned}$$
